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Orthogonally additive functions modulo a discrete subgroup

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Summary. Under appropriate conditions on the abelian groups G and H and the orthogonality $\perp \subset G^2$ we prove that a function $f : G \to H$ continuous at a point is orthogonally additive modulo a discrete subgroup K if and only if there exist a unique continuous additive function $a : G \to H$ and a unique continuous biadditive and symmetric function $b : G \times G \to H$ such that $f(x) - b(x,x) - a(x) \in K$ for $x \in G$ and $b(x,y) = 0$ for $x, y \in G$ such that $x \perp y$.

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In this paper we work with the following orthogonality proposed by K. Baron and P. Volkmann in [4]:

Let G be a group such that the mapping

$$x \mapsto 2x, \quad x \in G, \tag{1}$$

is a bijection onto the group G. A relation $\perp \subset G^2$ is called *orthogonality* if it satisfies the following two conditions:

(O) $0 \perp 0$; and from $x \perp y$ the relations $-x \perp -y$, $\frac{x}{2} \perp \frac{y}{2}$ follow.

(P) If an orthogonally additive function from G to an abelian group is odd, then it is additive; if it is even, then it is quadratic.

According to Theorems 5 and 6 from [7] the orthogonality considered by J. Rätz in [7] satisfies both (O) and (P).

Throughout this paper for a subset U of a given group and for $n \in \mathbb{N}$ the symbol nU denotes the set $\{nx : x \in U\}$.

Our main result reads as follows:

Theorem 1. *Assume that G is an abelian topological group such that the mapping* (1) *is a homeomorphism and the following condition holds:*

(H) *every neighbourhood of zero in G contains a neighbourhood U of zero such that*

$$U \subset 2U \tag{2}$$

and

$$G = \bigcup \{2^n U : \ n \in \mathbb{N}\}. \tag{3}$$

Assume $\perp \subset G^2$ is an orthogonality, H is an abelian topological group and K is a discrete subgroup of H. Then a function $f : G \to H$ continuous at a point satisfies

$$f(x+y) - f(x) - f(y) \in K \quad \text{for } x, y \in G \text{ such that } x \perp y \tag{4}$$

if and only if there exist a continuous additive function $a : G \to H$ and a continuous biadditive and symmetric function $b : G \times G \to H$ such that

$$f(x) - b(x,x) - a(x) \in K \quad \text{for } x \in G \tag{5}$$

and

$$b(x,y) = 0 \quad \text{for } x, y \in G \text{ such that } x \perp y. \tag{6}$$

Moreover, the functions a and b are uniquely determined.

Note that this theorem generalizes Theorem 2.9 from [6] and, in view of Theorem 9 from [7] and Theorem 4.2 from [3], also implies the result obtained in [1].

The proof of Theorem 1 will be presented after some lemmas. The first three lemmas and Lemma 4(i) are very similar to some results from [2], [6] and [5], but for the reader's convenience we formulate them explicitly; however, we omit their proofs. Note that Lemma 1(ii) [6, Lemma 2.3] is applied in the proof of Lemma 2 [6, Proposition 2.4], Lemma 1(i) [2, Lemma 1] and Lemma 2 in the proof of Lemma 3 [2, Theorem 3; 6, Theorem 2.6] and Lemma 3 in the proof of Lemma 4. Our Lemma 4(ii) can be proved in the same way as Lemma 4(i) [5, Lemma 4], so we also omit the proof.

Lemma 1. *Assume that G is an abelian group such that* (1) *is a bijection onto G, H is an abelian group and $U \subset G$ is a set with properties* (2) *and* (3)*.*

(i) *If $f : U \to H$ satisfies*

$$f(x+y) = f(x) + f(y) \quad \text{for } x, y \in U \text{ with } x+y \in U,$$

then it has a unique extension to an additive mapping of G into H.

(ii) *If $f : U \to H$ satisfies*

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \quad \text{for } x, y \in U \text{ with } x+y, x-y \in U$$

and $f(0) = 0$, then it has a unique extension to a quadratic mapping of G into H.

Lemma 2. *Assume that G is an abelian group such that* (1) *is a bijection onto G, H is an abelian group, K is a subgroup of H, $U \subset G$ is a set with properties* (2) *and* (3) *and W is a subset of H such that*

$$0 \in W, \quad W = -W \quad \text{and} \quad (W + W + W + W + W + W) \cap K = \{0\}.$$

If $f : G \to H$ satisfies

$$f(U) - f(0) \subset K + W$$

and

$$f(x + y) + f(x - y) - 2f(x) - 2f(y) \in K \quad \text{for } x, y \in G, \tag{7}$$

then $2f(0) \in K$ and there exists a quadratic function $q : G \to H$ such that

$$f(x) - q(x) - f(0) \in K \quad \text{for } x \in G, \tag{8}$$

$q(0) = 0$ and $q(U) \subset W$.

Lemma 3. *Assume that G is an abelian topological group such that* (1) *is a homeomorphism and* (H) *holds, H is an abelian topological group and K is a discrete subgroup of H.*

(i) *If $f : G \to H$ is continuous at zero and*

$$f(x + y) - f(x) - f(y) \in K \quad \text{for } x, y \in G,$$

then there exists a continuous additive function $a : G \to H$ such that

$$f(x) - a(x) \in K \quad \text{for } x \in G.$$

(ii) *If a function $f : G \to H$ continuous at zero satisfies* (7)*, then there exists a unique quadratic function $q : G \to H$ continuous at zero such that $q(0) = 0$ and* (8) *holds.*

In the rest of this paper we consider for an abelian topological group H and a subgroup K of H, the quotient group H/K with the quotient topology:

$$\{W \subset H/K : \ p^{-1}(W) \text{ is an open subset of } H\},$$

where $p : H \to H/K$ is the canonical mapping: $p(x) = x + K$.

Lemma 4. *Assume that G is an abelian topological group such that* (1) *is a homeomorphism and* (H) *holds, H is an abelian topological group and K is a discrete subgroup of H.*

(i) *If $A : G \to H/K$ is a continuous additive function, then there exists a continuous additive function $a : G \to H$ such that*

$$a(x) \in A(x) \quad \text{for } x \in G.$$

(ii) *If $Q : G \to H/K$ is a function which is continuous at zero and $Q(0) = K$, then there exists a continuous at zero quadratic function $q : G \to H$ such that $q(0) = 0$ and*

$$q(x) \in Q(x) \quad \text{for } x \in G.$$

The proof of the next lemma was kindly communicated to me by K. Baron.

Lemma 5. *Assume that G is an abelian topological group such that* (1) *is a homeomorphism and* (H) *holds and H is an abelian topological group. If a function $b : G \times G \to H$ is biadditive and continuous at $(0, 0)$, then it is continuous.*

Proof. First we prove that $b(x, \cdot)$ is continuous at zero for every $x \in G$. Take $x_0 \in G$ and a neighbourhood $W \subset H$ of zero. It follows from the continuity at zero of b and from (H) that there exists a neighbourhood $U \subset G$ of zero such that (3) and

$$b(U \times U) \subset W$$

hold. Consequently $x_0 = 2^n u_0$ with an $n \in \mathbb{N}$ and a $u_0 \in U$, and for $u \in U$ we have

$$b(x_0, 2^{-n}u) = b(2^n u_0, 2^{-n}u) = 2^n b(u_0, 2^{-n}u) = b(u_0, u) \in W.$$

Hence

$$b(x_0, 2^{-n}U) \subset W,$$

which shows that $b(x_0, \cdot)$ is continuous at zero. Clearly, the same concerns $b(\cdot, y_0)$ for every $y_0 \in G$. To finish the proof it is enough to observe now that

$$b(x, y) - b(x_0, y_0) = b(x - x_0, y_0) + b(x - x_0, y - y_0) + b(x_0, y - y_0)$$

holds for $x, y, x_0, y_0 \in G$. □

Our last lemma generalizes Theorem 4.3 from [3].

Lemma 6. *Assume that G is an abelian topological group such that* (1) *is a homeomorphism and* (H) *holds, $\perp \subset G^2$ is an orthogonality and H is an abelian topological group. If an orthogonally additive function $f : G \to H$ is continuous at some point, then it is continuous; more precisely, it is of the form*

$$f(x) = a(x) + b(x, x) \quad \text{for } x \in G, \tag{9}$$

where $a : G \to H$ is a continuous additive function, $b : G \times G \to H$ is a continuous biadditive and symmetric function and (6) *holds.*

Proof. According to Theorem 1 from [4] the function f has form (9), where $a : G \to H$ is additive, $b : G \times G \to H$ is biadditive, symmetric and satisfies (6); moreover,

$$b(x, y) = 2\Big(f\Big(\frac{x + y}{4}\Big) + f\Big(\frac{-x - y}{4}\Big) - f\Big(\frac{x - y}{4}\Big) - f\Big(\frac{-x + y}{4}\Big)\Big) \quad \text{for } x, y \in G. \tag{10}$$

Let $x_0 \in G$ be a continuity point of f. It follows from (9) that

$$f(x + x_0) - f(x) - f(x_0) = 2b(x, x_0) \quad \text{for } x \in G,$$

whence continuity at zero of $f + 2b(\cdot, x_0)$ follows. Consequently also the function

$$x \mapsto f(-x) + 2b(-x, x_0), \quad x \in G,$$

is continuous at zero. Summing up those two functions we get continuity at zero of

$$x \mapsto f(x) + f(-x), \quad x \in G.$$

Since (1) is a homeomorphism, this jointly with (10) gives continuity at $(0, 0)$ of b and applying Lemma 5 we see that b is continuous (at each point of $G \times G$). Hence and from (9) continuity of a (at x_0 and, consequently, everywhere) follows. This ends the proof. □

Proof of Theorem 1. The proof of the "if" part is easy, so we omit it. The "only if" part is divided into Parts I and II.

Part I. Assume that f satisfies (4) and define the function $\hat{f} : G \to H/K$ by the formula

$$\hat{f} = p \circ f.$$

Clearly $\hat{f}$ is continuous at a point, and (4) implies that $\hat{f}$ is orthogonally additive. According to Lemma 6 there exist a continuous additive function $\hat{a} : G \to H/K$ and a continuous quadratic function $\hat{q} : G \to H/K$ such that $\hat{q}(0) = K$ and

$$\hat{f}(x) = \hat{a}(x) + \hat{q}(x) \quad \text{for } x \in G.$$

By Lemma 4 we get a continuous additive function $a : G \to H$ and a quadratic function $q : G \to H$ continuous at zero such that $q(0) = 0$,

$$p \circ a = \hat{a} \quad \text{and} \quad p \circ q = \hat{q}.$$

Consequently, $f(x) - q(x) - a(x) + K = \hat{f}(x) - \hat{q}(x) - \hat{a}(x) = K$, i.e.,

$$f(x) - q(x) - a(x) \in K \quad \text{for } x \in G. \tag{11}$$

It follows from Lemma 2 from [4] that q has the form

$$q(x) = b(x, x) \quad \text{for } x \in G, \tag{12}$$

where $b : G \times G \to H$ is biadditive, symmetric and continuous at $(0, 0)$. Applying Lemma 5 we see that b is continuous.

Part II. Now we prove that q is orthogonally additive and that (6) holds.

Since K is discrete, there exists a neighbourhood $W \subset H$ of zero such that

$$K \cap W = \{0\}.$$

Let $W_0 \subset H$ be a symmetric neighbourhood of zero with

$$W_0 + W_0 + W_0 \subset W$$

and $U \subset G$ be a neighbourhood of zero such that $q(U) \subset W_0$, (2) and (3) hold.

Take $x, y \in G$ with $x \perp y$ and, making use of (3) and (2), choose an $n \in \mathbb{N}$ such that

$$2^{-n}x,\ 2^{-n}y,\ 2^{-n}(x + y) \in U.$$

Then

$$q(2^{-n}(x+y)) - q(2^{-n}x) - q(2^{-n}y) \in W_0 - W_0 - W_0 \subset W.$$

On the other hand, by (11) and (4),

$$q(2^{-n}(x+y)) - q(2^{-n}x) - q(2^{-n}y) \in f(2^{-n}(x+y)) - f(2^{-n}x) - f(2^{-n}y) + K = K.$$

Consequently,

$$q(2^{-n}(x+y)) - q(2^{-n}x) - q(2^{-n}y) = 0.$$

Moreover, by (12),

$$q(2^k z) = 2^{2k} q(z) \quad \text{for } z \in G \text{ and } k \in \mathbb{N}.$$

This yields

$$q(x+y) - q(x) - q(y) = 2^{2n}(q(2^{-n}(x+y)) - q(2^{-n}x) - q(2^{-n}y)) = 0$$

and, as $\frac{x}{2}$ and $\frac{y}{2}$ are also orthogonal,

$$b(x,y) = 4b\Big(\frac{x}{2}, \frac{y}{2}\Big) = 2\Big(q\Big(\frac{x}{2} + \frac{y}{2}\Big) - q\Big(\frac{x}{2}\Big) - q\Big(\frac{y}{2}\Big)\Big) = 0.$$

Part III: Uniqueness. Suppose $a_1 : G \to H$ is additive and continuous, $b_1 : G \times G \to H$ is biadditive, symmetric and continuous, and

$$f(x) - b_1(x,x) - a_1(x) \in K \quad \text{for } x \in G. \tag{13}$$

Putting

$$a_0 = a - a_1, \quad b_0 = b - b_1,$$

we get in view of (5) and (13)

$$a_0(x) + b_0(x,x) \in K \quad \text{for } x \in G, \tag{14}$$

which jointly with additivity of a_0 and biadditivity of b_0 gives

$$a_0(2x) = (a_0(x) + b_0(x,x)) - (a_0(-x) + b_0(-x,-x)) \in K$$

for $x \in G$. Consequently, since (1) is a bijection, $a_0(G) \subset K$. Hence, taking into account that K is discrete and a_0 is continuous and vanishes at zero, we infer that a_0 vanishes on a neighbourhood of zero and making use of (H) we see that a_0 vanishes everywhere. Thus $a_1 = a$ and (14) takes the form

$$b_0(x,x) \in K \quad \text{for } x \in G.$$

Reasoning as above we show that

$$b_0(x,x) = 0 \quad \text{for } x \in G,$$

whence

$$2b_0(x,y) = b_0(x+y, x+y) - b_0(x,x) - b_0(y,y) = 0$$

for $x, y \in G$ and, consequently,

$$b_0(x,y) = 4b_0\Big(\frac{x}{2}, \frac{y}{2}\Big) = 0$$

for $x, y \in G$, which means that $b_1 = b$. □

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