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MEASURABLE ORTHOGONALLY ADDITIVE FUNCTIONS MODULO A DISCRETE SUBGROUP*

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Abstract. Under appropriate conditions on Abelian topological groups G and H, an orthogonality $\perp \subset G^2$ and a σ-algebra $\mathfrak{M}$ of subsets of G we decompose an $\mathfrak{M}$-measurable function $f : G \to H$ which is orthogonally additive modulo a discrete subgroup K of H into its continuous additive and continuous quadratic part (modulo K).

1. Introduction

Throughout all the paper G and H are Abelian topological groups, K is a discrete subgroup of H.

Following K. Baron and P. Volkmann [2], in the case when G is uniquely 2-divisible, a relation $\perp \subset G^2$ is called *orthogonality* if it satisfies the following two conditions:

(O) $\quad 0 \perp 0$; and from $x \perp y$ the relations $-x \perp -y$, $\frac{x}{2} \perp \frac{y}{2}$ follow.

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(P) $\begin{cases}\text{If an orthogonally additive function from } G \text{ to an Abelian group is} \\ \text{odd, then it is additive; if it is even, then it is quadratic.}\end{cases}$

For instance, the orthogonality considered by J. Rätz in [13] fulfils both (O) and (P), according to Theorems 5 and 6 therein. For further examples the reader is referred to [2].

All along we assume that $\mathfrak{M}$ is a σ-algebra and $\mathfrak{I}$ is a proper σ-ideal of subsets of G which fulfil the condition:

$$0 \in \operatorname{Int}(A - A), \text{ if } A \in \mathfrak{M} \setminus \mathfrak{I}. \tag{S}$$

We deal with the problem: under what assumptions an $\mathfrak{M}$-measurable mapping $f : G \to H$ which is orthogonally additive modulo K, i.e.

$$f(x + y) - f(x) - f(y) \in K \quad \text{for} \quad x, y \in G \quad \text{such that} \quad x \perp y, \tag{1}$$

admits a factorization of the type

$$f(x) - b(x, x) - a(x) \in K \quad \text{for} \quad x \in G \tag{2}$$

with a continuous additive $a : G \to H$ and a separately/jointly continuous biadditive $b : G \times G \to H$?

The main aim of this paper is to establish representation (2) with a *jointly* continuous biadditive function b. This is done in the next section under some reasonable assumptions (on G or $\mathfrak{M}$). In the third section we obtain this decomposition with a separately continuous b under somewhat weaker conditions.

2. Factorization with a jointly continuous biadditive term

The first lemma is a kind of folklore and has been established in special cases when $\mathfrak{M}$ is the σ-algebra of subsets having the Baire property or being Christensen measurable. In both cases the key property is condition (S), where $\mathfrak{I}$ is the family of meager or Christensen zero subsets of G, respectively (see [12, Theorem 9.9] and [8, Theorem 2] with [10]). For the proof of this lemma see e.g. [12, Theorem 9.10].

LEMMA 1. *Every $\mathfrak{M}$-measurable homomorphism from G into a separable topological group is continuous.*

LEMMA 2. *Let X be a topological space with a countable base. If the functions $f, g : G \to X$ are $\mathfrak{M}$-measurable, then so is the function $(f, g) : G \to X \times X$. Consequently, if Y is a topological space and $\varphi : X \times X \to Y$ is a Borel function, then $\varphi(f, g)$ is $\mathfrak{M}$-measurable.*

PROOF. It is enough to observe that if $\mathcal{B}$ is a countable base of X, then $\{V \times W : V, W \in \mathcal{B}\}$ is a countable base of $X \times X$. $\square$

LEMMA 3. *Assume H is separable metric and at least one of the conditions holds:*

(i) *G is a first countable Baire group;*

(ii) *G is separable metric;*

(iii) *G is metric and $\mathfrak{M}$ contains all Borel subsets of G.*

If a biadditive function $b : G \times G \to H$ has $\mathfrak{M}$-measurable sections $b(x,\cdot)$, $b(\cdot,y)$ for all $x, y \in G$, then b is continuous.

PROOF. If G is a first countable Baire group, then [9, Proposition 2.3] implies that (G, G, H) forms a Namioka–Troallic triple. Our assertion then follows from the fact that the sections of b being $\mathfrak{M}$-measurable are, according to Lemma 1, continuous, and from the H. R. Ebrahimi-Vishki result [9, Theorem 3.2].

Let d_G, d_H stand for invariant metrics for G, H, respectively (cf. [11, Theorem 8.3]), $B(r) = \{ z \in G : d_G(z,0) \leqq r \}$ for positive $r \in \mathbb{R}$ and

$$F_{n,k} = \Big\{ x \in G : d_H\big(b(x,u), b(x,v)\big) \leqq 2^{-n} \text{ for all } u, v \in B\big(2^{-k}\big) \Big\}$$

for $n, k \in N$. By Lemma 1, the sections $b(\cdot,u)$ are continuous for $u \in G$, whence $F_{n,k}$ are closed for $n, k \in \mathbb{N}$. Consequently, in case (iii) we have

$$F_{n,k} \in \mathfrak{M} \quad \text{for } n, k \in \mathbb{N}. \tag{3}$$

To show that (3) holds also in case (ii) for every $k \in \mathbb{N}$ consider a countable and dense subset D_k of $B(2^{-k})$. Then, due to continuity of $b(x,\cdot)$ for $x \in G$, we have

$$F_{n,k} = \bigcap_{(u,v) \in D_k} \Big\{ x \in G : d_H\big(b(x,u), b(x,v)\big) \leqq 2^{-n} \Big\} \quad \text{for } n, k \in \mathbb{N}.$$

Moreover, as follows from Lemma 2, the mapping $G \ni x \mapsto d_H\big(b(x,u), b(x,v)\big)$ is $\mathfrak{M}$-measurable for $u, v \in G$. Hence we have (3) also in case (ii).

Because of the continuity of $b(x,\cdot)$, we have

$$G = \bigcup_{k \in \mathbb{N}} F_{n,k} \quad \text{for } n \in \mathbb{N}.$$

Consequently, if $n \in \mathbb{N}$, then $F_{n,k(n)} \in \mathfrak{M} \setminus \mathfrak{I}$ for at least one $k(n) \in \mathbb{N}$. This fact, jointly with condition (S), yield

$$0 \in \operatorname{Int}\big(F_{n,k(n)} - F_{n,k(n)}\big). \tag{4}$$

On the other hand, if $k, n \in \mathbb{N}$, $n \geqq 2$, then for all $x, x' \in F_{n,k}$ and all $u, v \in B(2^{-k})$ we have

$$\begin{aligned} d_H\big(b(x-x',u), b(x-x',v)\big) &= d_H\big(b(x,u)-b(x',u), b(x,v)-b(x',v)\big) \\ &= d_H\big(b(x,u), b(x,v)+b(x',u-v)\big) \\ &\leqq d_H\big(b(x,u), b(x,v)\big) + d_H\big(b(x,v), b(x,v)+b(x',u-v)\big) \\ &= d_H\big(b(x,u), b(x,v)\big) + d_H\big(b(x',v), b(x',u)\big) \leqq 2^{-(n-1)}, \end{aligned}$$

which shows that $F_{n,k} - F_{n,k} \subset F_{n-1,k}$. Combining this with (4) we infer that for all $n \in \mathbb{N}$ there is $k(n) \in \mathbb{N}$ and $r(n) > 0$ such that

$$(5) \qquad d_H\big(b(x,u), b(x,v)\big) \leqq 2^{-n} \quad \text{for } x \in B\big(r(n)\big) \text{ and } u, v \in B\big(2^{-k(n)}\big).$$

Fix any (x,u) and (x',v) from $B\big(\frac{1}{2}r(n)\big) \times B\big(2^{-k(n)}\big)$. Then

$$x - x' \in B\big(0, r(n)\big)$$

and (5) yields

$$\begin{aligned} d_H\big(b(x,u), b(x',v)\big) &\leqq d_H\big(b(x,u), b(x,v)\big) + d_H\big(b(x,v), b(x',v)\big) \\ &\leqq 2^{-n} + d_H\big(b(x-x',v), 0\big) \\ &= 2^{-n} + d_H\big(b(x-x',v), b(x-x',0)\big) \leqq 2^{-(n-1)}. \end{aligned}$$

This proves the continuity of b at $(0,0)$. Since

$$b(x,y) - b(x_0,y_0) = b(x-x_0,y_0) + b(x-x_0,y-y_0) + b(x_0,y-y_0)$$

for $x, y \in G$ and $b(\cdot, y_0)$, $b(x_0, \cdot)$ are continuous, b is therefore continuous at every point $(x_0,y_0) \in G \times G$. $\square$

Note that in the special case when $\mathfrak{M}$ consists of all sets with the Baire property, the assumption that G is Baire, or equivalently G is non-meager (see e.g. [12, Proposition 9.8]), corresponds to our hypothesis $G \notin \mathfrak{I}$.

A key role in the above proof is played by condition (S). Even in the case when G is a real separable normed space and $\mathfrak{M}$ is the σ-algebra of its Borel subsets, a suitable σ-ideal $\mathfrak{I}$ which satisfies (S) does not have to exist. Consider, for instance, the space of all real polynomials of one variable with the norm $\|f\| = \int_0^1 \big|f(t)\big|\,dt$ and the bilinear functional $B(f,g) = \int_0^1 f(t)g(t)\,dt$ which is separately but not jointly continuous. In view of our last lemma, such a space does not admit a σ-ideal $\mathfrak{I}$ which would fulfil condition (S). For the essentiality of the above assumptions cf. also Example 3.3 in [9].

LEMMA 4. *If H is separable metric, then the quotient group H/K is an Abelian separable metric group.*

PROOF. Since K is closed in H, the group H/K is Hausdorff (see [11, Theorem 5.21]). Because H has a countable base, so has also H/K. In the light of the Birkhoff–Kakutani theorem [11, Theorem 8.3], H/K is thus metrizable. Separability follows again from the existence of a countable base. $\square$

Now we are prepared to proceed to our main result. The technical assumptions appearing below have been already considered (see [7], [3], [6] and [14]). In the last section we present a counterexample showing that condition (G2) is essential.

THEOREM 1. *Assume H is separable metric,*

(G1) *the mapping $G \ni x \mapsto 2x$ is a homeomorphism,*

(G2) *every neighbourhood of zero in G contains a zero neighbourhood U such that*

$$U \subset 2U \quad \text{and} \quad G = \bigcup \{2^n U : n \in \mathbb{N}\}, \tag{6}$$

(G3) *either G is a first countable Baire group, or G is metric separable, or G is metric and $\mathfrak{M}$ contains all Borel subsets of G,*

(G4) *$x \pm 2A \in \mathfrak{M}$ for all $x \in G$ and $A \in \mathfrak{M}$.*

Then an $\mathfrak{M}$-measurable function $f : G \to H$ satisfies (1) *if and only if there exist a continuous additive function $a : G \to H$ and a continuous biadditive symmetric function $b : G \times G \to H$ such that the factorization* (2) *is valid, and*

$$b(x, y) = 0 \quad \text{for} \quad x, y \in G \quad \text{such that} \quad x \perp y; \tag{7}$$

moreover, the functions a and b are uniquely determined.

PROOF. Define $\hat{f} : G \to H/K$ as $\hat{f} = p \circ f$ where p stands for the canonical projection. Condition (1) yields the orthogonal additivity of $\hat{f}$. By [2, Theorem 1], there exist an additive function $\hat{a} : G \to H/K$ and a quadratic function $\hat{q} : G \to H/K$ such that $\hat{f} = \hat{a} + \hat{q}$. Moreover the function $\hat{a}$ is defined by the formula

$$\hat{a}(x) = \hat{f}\left(\frac{x}{2}\right) - \hat{f}\left(-\frac{x}{2}\right)$$

and $\hat{q}(x) = \hat{b}(x, x)$, $x \in G$, with a biadditive and symmetric function $\hat{b} : G \times G \to H/K$ given by

$$\hat{b}(x, y) = 2\left[\hat{f}\left(\frac{x+y}{4}\right) + \hat{f}\left(\frac{-x-y}{4}\right) - \hat{f}\left(\frac{x-y}{4}\right) - \hat{f}\left(\frac{-x+y}{4}\right)\right].$$

The above equalities, jointly with $\mathfrak{M}$-measurability of $\hat{f}$, condition (G4) and Lemma 2, imply the $\mathfrak{M}$-measurability of $\hat{a}$ and the sections $\hat{b}(x,\cdot)$ for every $x \in G$. By Lemmas 4, 1 and 3, the functions $\hat{a}$ and $\hat{b}$ are continuous.

According to [14, Lemma 4] there exist a continuous additive function $a : G \to H$ and a continuous at zero quadratic function $q : G \to H$ such that $q(0) = 0$ and $p \circ a = \hat{a}$, $p \circ q = \hat{q}$. Hence $f(x) - q(x) - a(x) \in K$ for $x \in G$. As in the proof of [14, Theorem 1] we recall [2, Lemma 2] and [14, Lemma 5] to obtain $q(x) = b(x,x)$ with a continuous biadditive symmetric function $b : G \times G \to H$. To finish the proof of the "only if" part it remains to apply Lemma 5 given below.

The proof of the "if" part is a simple verification. □

LEMMA 5. *Assume* (G1) *and* (G2). *Let the functions $a_1, a_2 : G \to H$ be continuous additive and let $b_1, b_2 : G \times G \to H$ be biadditive symmetric and continuous in each variable.*

(i) *If* $\big(a_1(x) + b_1(x,x)\big) - \big(a_2(x) + b_2(x,x)\big) \in K$ *for $x \in G$, then $a_1 = a_2$ and $b_1 = b_2$.*

(ii) *If $b_1(x,y) \in K$ for $x, y \in G$ such that $x \perp y$, then $b_1(x,y) = 0$ for $x, y \in G$ such that $x \perp y$.*

PROOF. (i) Let $a := a_1 - a_2$, $b := b_1 - b_2$. For $x \in G$ we have $a(x) + b(x,x) \in K$. Hence

$$a(2x) = \big(a(x) + b(x,x)\big) - \big(a(-x) + b(-x,-x)\big) \in K,$$

which implies $a(G) \subset K$. Now, condition (G2) guarantees that the function a, being continuous and additive, is constantly equal to zero.

We have just obtained that $b(x,x) \in K$ for $x \in G$, thus

$$b(x,2y) = 2b(x,y) = b(x+y,x+y) - b(x,x) - b(y,y) \in K \quad \text{for} \quad x,y \in G.$$

Arguing as above we infer that the section $b(\cdot,2y)$ is constantly equal to zero for every $y \in G$, so $b = 0$.

(ii) Fix $x, y \in G$ such that $x \perp y$. Choose zero neighbourhoods $W \subset G$ such that $K \cap W = \{0\}$ and $U \subset G$ such that

$$b(U,y) \subset W \text{ and } G = \bigcup\{2^n U : n \in \mathbb{N}\}.$$

For some $n \in \mathbb{N}$ we have $x \in 2^n U$, whence $b\big(\frac{x}{2^n}, y\big) \in W$. Plainly, $2^{-n}x \perp 2^{-n}y$, which implies

$$b\left(\frac{x}{2^n}, y\right) = 2^n b\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \in K.$$

Consequently, $b(2^{-n}x, y) = 0$ and

$$b(x,y) = 2^n b\left(\frac{x}{2^n}, y\right) = 0,$$

as desired. □

As a consequence of Theorem 1 we obtain the following result.

COROLLARY 1. *Assume H is separable metric and* (G1), (G2) *hold. If either G is a first countable Baire group and $f: G \to H$ is Baire measurable, or G is a Polish group and $f: G \to H$ is Christensen measurable, then f satisfies* (1) *if and only if there exist a continuous additive function $a: G \to H$ and a continuous biadditive symmetric function $b: G \times G \to H$ such that* (2) *and* (7) *hold; moreover, the functions a and b are uniquely determined.*

Baire and Christensen measurable solutions of (1) have been already examined by J. Brzdęk in [4] for the orthogonality given by an inner product and in [5] for a more abstract orthogonality in linear topological spaces.

3. Factorization with a separately continuous biadditive term

Under weaker assumptions we obtain the factorization (2) with a separately continuous biadditive term only (as it is in [5, Theorem 1]).

THEOREM 2. *Assume* (G1), (G2), (G4) *and let H be separable metric. Then an $\mathfrak{M}$-measurable function $f: G \to H$ satisfies* (1) *if and only if there exist a continuous additive function $a: G \to H$ and a function $b: G \times G \to H$ biadditive symmetric and continuous in each variable such that the factorization* (2) *is valid and* (7) *holds; moreover, the functions a and b are uniquely determined.*

To get this result we argue as in the proof of Theorem 1 but without referring to Lemma 3 and applying the following Lemma 6 instead of [14, Lemma 4(ii)].

LEMMA 6. *Assume* (G1) *and* (G2). *If $\hat{b}: G \to H/K$ is biadditive, symmetric and continuous in each variable, then there exists a function $b: G \times G \to H$ biadditive, symmetric and continuous in each variable such that*

$$b(x,y) \in \hat{b}(x,y) \quad \text{for } (x,y) \in G \times G. \tag{8}$$

PROOF. It follows from [14, Lemma 4(i)] that there exists a function $b: G \times G \to H$ such that for every $y \in G$ the function $b(\cdot, y)$ is additive, continuous and (8) holds. To show that b is symmetric fix $x, y \in G$ and a neighbourhood W of zero in H with

$$(W + W - W) \cap K = \{0\}.$$

Since $b(\cdot,y)^{-1}(W) \cap b(\cdot,2y)^{-1}(W) \cap b(\cdot,x)^{-1}(W)$ is a neighbourhood of zero, it follows from (G2) that there exists a zero neighbourhood U such that

$$U \subset b(\cdot,y)^{-1}(W) \cap b(\cdot,2y)^{-1}(W) \cap b(\cdot,x)^{-1}(W)$$

and (6) holds. In particular, $x = 2^n u_1$ and $y = 2^n u_2$ for some $n \in \mathbb{N}$ and $u_1, u_2 \in U$. Moreover,

$$2b(u_1, y) - b(u_1, 2y) \in (2W - W) \cap \big(2\hat{b}(u_1, y) - \hat{b}(u_1, 2y)\big)$$
$$= (2W - W) \cap K = \{0\},$$

whence $2b(u_1, y) = b(u_1, 2y)$ and, consequently,

$$2b(x, y) = 2b(2^n u_1, y) = 2^n \cdot 2b(u_1, y) = 2^n b(u_1, 2y) = b(x, 2y).$$

Now, having the equality $b(x, 2y) = 2b(x, y)$ for any $x, y \in G$ we see that

$$b(x, u_2) = b(2^n u_1, u_2) = b(u_1, 2^n u_2) = b(u_1, y) \in W,$$

whence

$$b(x, u_2) - b(u_2, x) \in (W - W) \cap \big(\hat{b}(x, u_2) - \hat{b}(u_2, x)\big) = (W - W) \cap K = \{0\}$$

and

$$b(x, y) = b(x, 2^n u_2) = 2^n b(x, u_2) = 2^n b(u_2, x) = b(2^n u_2, x) = b(y, x). \quad \square$$

As a consequence we obtain a corollary asserting that if G is Baire and we consider the Baire measurability, then we do not need to assume the first countability of G in order to get the desired factorization with a separately continuous biadditive term only (cf. Corollary 1).

COROLLARY 2. *Assume H is separable metric and* (G1), (G2) *hold. If G is Baire and $f : G \to H$ is Baire measurable, then f satisfies* (1) *if and only if there exist a continuous additive function $a : G \to H$ and a function $b : G \times G \to H$ biadditive symmetric and continuous in each variable such that* (2) *and* (7) *hold; moreover, the functions a and b are uniquely determined.*

If we take $\perp = G^2$, then Theorem 2 gives us Corollary 3 below. Of course, again it leads to another conclusions in the case when the measurability that we consider is Baire or Christensen.

COROLLARY 3. *Assume* (G1), (G2), (G4) *and let H be separable metric. Then an $\mathfrak{M}$-measurable function $f : G \to H$ satisfies*

$$f(x + y) - f(x) - f(y) \in K \quad \text{for} \quad x, y \in G$$

if and only if there exists a (unique) continuous additive function $a : G \to H$ such that

$$f(x) - a(x) \in K \quad \text{for} \quad x \in G.$$

4. A counterexample

Hypothesis (G2) is supposed to be a substitute for the condition that every zero neighbourhood is absorbing – the condition which we dispose of in linear topological spaces. The following example shows that we cannot run too far away from this linear topological structure. Although for the simplest counterexample we may consider $(\mathbb{R},+)$ with the discrete topology, we present a more interesting one. Our aim is to demonstrate that the validity of all of the assumptions, just with the exception of (G2), does not guarantee the factorization (2) even if the domain is a "nice" structure with a non-discrete topology.

Let $\mathbb{R}^{\mathbb{N}}$ stand for the group of all real sequences (with the ordinary addition). In this group we introduce the so called *Krull topology,* the Tychonov (product) topology with the discrete topology in $\mathbb{R}$. Observe that we obtain in this manner an Abelian topological group metrizable by a complete metric. In particular, it is a Baire group. Note also that the family $\{V_I : I \in \mathcal{F}\}$, where

$$\mathcal{F} := \{I \subset \mathbb{N} : \operatorname{card} I < \aleph_0\}$$

and

$$V_I := \left\{ (x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : x_i = 0 \text{ for } i \in I \right\} \quad \text{for} \quad I \in \mathcal{F}$$

is a zero neighbourhood basis.

Clearly, $\mathbb{R}^{\mathbb{N}}$ is uniquely 2-divisible (it is even a real linear space) and the orthogonality $\perp$ defined as $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ fulfils both (O) and (P). Obviously, the mapping $\mathbb{R}^{\mathbb{N}} \ni x \mapsto 2x$ is a homeomorphism. However, since V_I is a subgroup of $\mathbb{R}^{\mathbb{N}}$, we have

$$\bigcup\{nV_I : n \in \mathbb{N}\} = V_I \subsetneqq \mathbb{R}^{\mathbb{N}} \quad \text{for} \quad I \in \mathcal{F},\ I \neq \emptyset.$$

Let $\mathfrak{B}$ be the σ-algebra of all Borel subsets of $\mathbb{R}^{\mathbb{N}}$ and let $\mathfrak{J}$ be the (proper) σ-ideal of all meager subsets of $\mathbb{R}^{\mathbb{N}}$. The classical theorem of Pettis [12, Theorem 9.9] asserts that $0 \in \operatorname{Int}(A - A)$, whenever $A \in \mathfrak{B} \setminus \mathfrak{J}$.

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be any function fulfilling the congruence

$$\varphi(x+y) - \varphi(x) - \varphi(y) \in \mathbb{Z} \quad \text{for} \quad x, y \in \mathbb{R}$$

which is not a sum of an additive and a $\mathbb{Z}$-valued function (see [1, Remark 2] for a suitable example). Define $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ by the formula

$$f(x) = \varphi(x_1) \quad \text{for} \quad x = (x_n)_{n\in\mathbb{N}}.$$

Plainly, f is a continuous (hence Borel) solution of the congruence

$$f(x+y) - f(x) - f(y) \in \mathbb{Z} \quad \text{for} \quad x, y \in \mathbb{R}^{\mathbb{N}}.$$

Now, suppose that $f(x) - b(x,x) - a(x) \in \mathbb{Z}$ for $x \in \mathbb{R}^{\mathbb{N}}$ with an additive function $a : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ and a function $b : \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ which fulfils (7). Since our orthogonality is the trivial one, we have $b = 0$ and hence

$$f(x) - a(x) \in \mathbb{Z} \quad \text{for} \quad x \in \mathbb{R}^{\mathbb{N}}. \tag{9}$$

Defining $\alpha : \mathbb{R} \to \mathbb{R}$ by $\alpha(x) = a(x, 0, 0, \ldots)$ we see that it is additive and (9) implies that

$$\varphi(x) - \alpha(x) = f(x, 0, 0, \ldots) - a(x, 0, 0, \ldots) \in \mathbb{Z} \quad \text{for} \quad x \in \mathbb{R},$$

contrary to the choice of φ.

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