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ORTHOGONALLY PEXIDER FUNCTIONS MODULO A DISCRETE SUBGROUP

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Abstract. Under appropriate conditions on abelian topological groups G and H, an orthogonality $\perp \subset G^2$ and a σ-algebra $\mathfrak{M}$ of subsets of G we prove that if at least one of the functions $f, g, h\colon G \to H$ satisfying

$$f(x+y) - g(x) - h(y) \in K \quad \text{for } x, y \in G \text{ such that } x \perp y,$$

where K is a discrete subgroup of H, is continuous at a point or $\mathfrak{M}$-measurable, then there exist: a continuous additive function $A\colon G \to H$, a continuous biadditive and symmetric function $B\colon G \times G \to H$ and constants $a, b \in H$ such that

$$\begin{cases} f(x) - B(x,x) - A(x) - a \in K, \\ g(x) - B(x,x) - A(x) - b \in K, \\ h(x) - B(x,x) - A(x) - a + b \in K \end{cases}$$

for $x \in G$ and

$$B(x,y) = 0 \quad \text{for } x, y \in G \text{ such that } x \perp y.$$

Let G and H be groups and $\perp \subset G^2$ an orthogonality. We say that a function $f\colon G \to H$ is orthogonally additive, if

$$f(x+y) = f(x) + f(y) \quad \text{for } x, y \in G \text{ such that } x \perp y.$$

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In the paper [3] J. Brzdęk considers the Rätz orthogonality (cf.[5]) and, under some assumptions, gives a description of orthogonally additive functions modulo a discrete subgroup, i.e. functions $f\colon G \to H$ such that

$$f(x+y) - f(x) - f(y) \in K \quad \text{for } x, y \in G \text{ such that } x \perp y,$$

where K is a discrete subgroup of H. In the papers [7] and [4] authors prove similar theorems (for continuous or measurable functions), but for the orthogonality defined by K. Baron and P. Volkmann in [2], which includes the Rätz orthogonality.

Now we would like to obtain some similar results for the Pexider difference instead of the Cauchy difference, i.e. we assume that functions $f, g, h\colon G \to H$ are orthogonally Pexider modulo a discrete subgroup, which means that they satisfy

$$f(x+y) - g(x) - h(x) \in K \quad \text{for } x, y \in G \text{ such that } x \perp y,$$

where K is a discrete subgroup of H. We start with the following result.

Lemma. *Let G be a groupoid with a neutral element, H an abelian group, K a subgroup of H. Let $\Delta \subset G \times G$ be a set with*

(1) $$(0, x), (x, 0) \in \Delta \quad \textit{for all } x \in G.$$

If functions $f, g, h\colon G \to H$ satisfy

(2) $$f(x+y) - g(x) - h(y) \in K \quad \textit{for } (x, y) \in \Delta,$$

then the following are true:

(a) *There are functions $k_1, l_1 \colon G \to K$, $\varphi_1 \colon G \to H$ and constants $a, b \in H$ such that*

$$\varphi_1(x+y) - \varphi_1(x) - \varphi_1(y) \in K \quad \textit{for } (x, y) \in \Delta$$

and

(3) $$\begin{cases} f(x) = \varphi_1(x) + a, \\ g(x) = \varphi_1(x) + k_1(x) + b, \\ h(x) = \varphi_1(x) - k_1(x) + l_1(x) + a - b \end{cases}$$

for all $x \in G$.

(b) *There are functions* $k_2, l_2\colon G \to K$, $\varphi_2\colon G \to H$ *and constants* $a, b \in H$ *such that*

$$\varphi_2(x+y) - \varphi_2(x) - \varphi_2(y) \in K \quad \text{for } (x,y) \in \Delta$$

and

$$\begin{cases} f(x) = \varphi_2(x) + k_2(x) + a, \\ g(x) = \varphi_2(x) + b, \\ h(x) = \varphi_2(x) + l_2(x) + a - b \end{cases}$$

for all $x \in G$.

(c) *There are functions* $k_3, l_3\colon G \to K$, $\varphi_3\colon G \to H$ *and constants* $a, b \in H$ *such that*

$$\varphi_3(x+y) - \varphi_3(x) - \varphi_3(y) \in K \quad \text{for } (x,y) \in \Delta$$

and

$$\begin{cases} f(x) = \varphi_3(x) + k_3(x) + a, \\ g(x) = \varphi_3(x) + l_3(x) + b, \\ h(x) = \varphi_3(x) + a - b \end{cases}$$

for all $x \in G$.

Moreover, each of assertions (a), (b), (c) *gives a complete description of solutions of* (2), *that is, every triple* (f, g, h), *being of one of the forms described above, is a solution of* (2).

Proof. Setting $y = 0$ in (2), by (1) we get

$$\mu(x) := f(x) - g(x) - h(0) \in K \quad \text{for } x \in G \tag{4}$$

and setting $x = 0$ we have

$$\nu(y) := f(y) - g(0) - h(y) \in K \quad \text{for } y \in G. \tag{5}$$

In particular,

$$f(0) - g(0) - h(0) \in K. \tag{6}$$

Denote $a = f(0)$, $b = g(0)$ and define $\varphi_i, k_i, l_i \colon G \to H$ for $i = 1, 2, 3$ by

$$
\begin{aligned}
&\varphi_1 = f - a, && k_1 = g - \varphi_1 - b, && l_1 = h + k_1 - \varphi_1 - a + b,\\
&\varphi_2 = g - b, && k_2 = f - \varphi_2 - a, && l_2 = h - \varphi_2 - a + b,\\
&\varphi_3 = h - a + b, && k_3 = f - \varphi_3 - a, && l_3 = g - \varphi_3 - b.
\end{aligned}
$$

Using (4), (5), (2) and (6) for every $(x, y) \in \Delta$ we get

$$
\begin{aligned}
\varphi_1(x+y) - \varphi_1(x) - \varphi_1(y) &= f(x+y) - a - f(x) + a - f(y) + a\\
&= f(x+y) - \mu(x) - g(x) - h(0) - \nu(y) - g(0) - h(y) + a \in K;\\
\varphi_2(x+y) - \varphi_2(x) - \varphi_2(y) &= g(x+y) - b - g(x) + b - g(y) + b\\
&= f(x+y) - \mu(x+y) - h(0) - g(x) + \mu(y) - f(y) + h(0) + b\\
&= f(x+y) - \mu(x+y) - g(x) + \mu(y) - \nu(y) - g(0) - h(y) + b \in K;\\
\varphi_3(x+y) - \varphi_3(x) - \varphi_3(y) &= h(x+y) - a + b - h(x) + a - b - h(y) + a - b\\
&= f(x+y) - g(0) - \nu(x+y) + \nu(x) - f(x) + g(0) - h(y) + a - b\\
&= f(x+y) - \nu(x+y) + \nu(x) - \mu(x) - g(x) - h(0) - h(y) + a - b,\\
&\in K.
\end{aligned}
$$

We also have

$$
\begin{aligned}
k_1(x) &= g(x) - f(x) + a - b = -\mu(x) - h(0) + a - b \in K,\\
k_2(x) &= f(x) - g(x) + b - a = \mu(x) + h(0) + b - a \in K,\\
k_3(x) &= f(x) - h(x) + a - b - a = \nu(x) + g(0) - b \in K,\\
l_1(x) &= h(x) + k_1(x) - f(x) + a - a + b = -\nu(x) - g(0) + k_1(x) + b \in K,\\
l_2(x) &= h(x) + k_2(x) - f(x) + a - a + b = -\nu(x) - g(0) + k_2(x) + b \in K,\\
l_3(x) &= g(x) + k_3(x) - f(x) + a - b = -\mu(x) - h(0) + k_3(x) + a - b \in K
\end{aligned}
$$

for $x \in G$. □

The part (b) of this lemma in the case when $\Delta = G^2$ was also obtained by K. Baron and PL. Kannappan in [1], even under some weaker assumptions. Some variations of (2) for functions with values in groupoids were studied by J. Sikorska in [6].

We work with the orthogonality proposed by K. Baron and P. Volkmann in [2], assuming additionally that the last condition in the following definition holds:

Let G be a group such that the mapping

$$x \mapsto 2x, \quad x \in G, \tag{7}$$

is a bijection onto the group G. A relation $\perp \subset G^2$ is called *orthogonality* if it satisfies the following three conditions:

(i) $0 \perp 0$; and from $x \perp y$ the relations $-x \perp -y$, $\frac{x}{2} \perp \frac{y}{2}$ follow.

(ii) If an orthogonally additive function from G to an abelian group is odd, then it is additive; if it is even, then it is quadratic.

(iii) $x \perp 0$ and $0 \perp x$ for every $x \in G$.

For a subset U of a given group and for $n \in \mathbb{N}$ the symbol nU denotes the set $\{nx : x \in U\}$.

THEOREM. *Assume G is an abelian topological group such that the mapping (7) is a homeomorphism and every neighbourhood of zero in G contains a neighbourhood U of zero such that*

$$U \subset 2U \quad \textit{and} \quad G = \bigcup\{2^n U : \ n \in \mathbb{N}\}. \tag{8}$$

Let $\perp \subset G^2$ be an orthogonality, H an abelian topological group and K a discrete subgroup of H. Assume that functions $f, g, h \colon G \to H$ satisfy

$$f(x+y) - g(x) - h(y) \in K \quad \textit{for } x, y \in G \textit{ such that } x \perp y. \tag{9}$$

(i) *If at least one of the functions f, g, h is continuous at a point, then there exist: a continuous additive function $A \colon G \to H$, a continuous biadditive and symmetric function $B \colon G \times G \to H$ and constants $a, b \in H$ such that*

$$\begin{cases} f(x) - B(x,x) - A(x) - a \in K, \\ g(x) - B(x,x) - A(x) - b \in K, \\ h(x) - B(x,x) - A(x) - a + b \in K \end{cases} \tag{10}$$

for $x \in G$ and

$$B(x,y) = 0 \quad \textit{for } x, y \in G \textit{ such that } x \perp y. \tag{11}$$

(ii) *Let $\mathfrak{M}$ be a σ-algebra of subsets of G such that*

$$x \pm 2A \in \mathfrak{M} \quad \textit{for all } x \in G \textit{ and } A \in \mathfrak{M} \tag{12}$$

and there is a proper σ-ideal $\mathfrak{I}$ of subsets of G with

$$0 \in \operatorname{Int}(A - A) \quad \textit{for } A \in \mathfrak{M} \setminus \mathfrak{I}. \tag{13}$$

Assume moreover that H is separable metric and the following condition (G) *is fulfilled:*

(G) *either G is a first countable Baire group, or G is metric separable, or G is metric and $\mathfrak{M}$ contains all Borel subsets of G.*

If at least one of the functions f, g, h is $\mathfrak{M}$-measurable, then there exist: a continuous additive function $A\colon G \to H$, a continuous biadditive and symmetric function $B\colon G \times G \to H$ and constants $a, b \in H$ such that (10) *and* (11) *hold.*

Moreover, each of assertions (i), (ii) *gives a complete description of solutions of* (9).

Proof. (i): Case 1. Assume that f is continuous at a point. Let $k_1, l_1 : G \to K$, $\varphi_1\colon G \to H$ be as in Lemma (a). Then the function φ_1 is continuous at a point. According to Theorem 1 from [7] we get a continuous additive function $A\colon G \to H$ and a continuous biadditive and symmetric function $B\colon G \times G \to H$ such that

$$\varphi_1(x) - B(x,x) - A(x) \in K \quad \text{for } x \in G$$

and (11) hold. Then, according to (3),

$$\begin{aligned} f(x) - B(x,x) - A(x) - a &= \varphi_1(x) + a - B(x,x) - A(x) - a \in K, \\ g(x) - B(x,x) - A(x) - b &= \varphi_1(x) + k_1(x) + b - B(x,x) - A(x) - b \in K, \\ h(x) - B(x,x) - A(x) - a + b &= \varphi_1(x) - k_1(x) + l_1(x) + a - b \\ &\qquad - B(x,x) - A(x) - a + b \in K \end{aligned}$$

for all $x \in G$.

Case 2. If the function g is continuous at a point then instead of Lemma (a) we use Lemma (b).

Case 3. If the function h is continuous at a point then we use Lemma (c).

(ii): If one of the functions f, g, h is $\mathfrak{M}$-measurable then we use Theorem 1 from [4] instead of Theorem 1 from [7]. □

For $\perp = G^2$ some special cases were obtained in [1] (cf. Corollaries 6 and 7 there).

If in the Theorem G is Baire and we consider the Baire measurability, then we do not need to assume the first countability of G in order to get the factorization with a separately continuous biadditive term only (cf. Corollary 2 in [4]).

Corollary 1. *Assume G is an abelian topological group such that the mapping* (7) *is a homeomorphism and every neighbourhood of zero in G contains a neighbourhood U of zero such that* (8) *holds. Let $\perp \subset G^2$ be an* *orthogonality, H an abelian separable metric group, K a discrete subgroup of H and functions $f, g, h\colon G \to H$ satisfy* (9)*. If G is Baire and at least one of the functions f, g, h is Baire measurable, then there exist: a continuous additive function $A\colon G \to H$, a function $B\colon G \times G \to H$ biadditive, symmetric and continuous in each variable, and constants $a, b \in H$ such that* (10) *and* (11) *hold.*

If we take $\perp = G^2$, then our Theorem gives us Corollary 2 below. It also leads to another conclusions in the case when we consider Baire or Christensen measurability.

COROLLARY 2. *Assume G is an abelian topological group such that the mapping* (7) *is a homeomorphism and every neighbourhood of zero in G contains a neighbourhood U of zero such that* (8) *holds. Let H be an abelian separable metric group, K a discrete subgroup of H, $\mathfrak{M}$ a σ-algebra of subsets of G satisfying* (12) *and such that there is a proper σ-ideal $\mathfrak{I}$ of subsets of G with property* (13)*. If functions $f, g, h\colon G \to H$ satisfy*

$$f(x+y) - g(x) - h(y) \in K \quad \textit{for } x, y \in G$$

and at least one of them is $\mathfrak{M}$-measurable, then there exist a continuous additive function $A\colon G \to H$ and constants $a, b \in H$ such that

$$\begin{cases} f(x) - A(x) - a \in K, \\ g(x) - A(x) - b \in K, \\ h(x) - A(x) - a + b \in K \end{cases}$$

for $x \in G$.

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