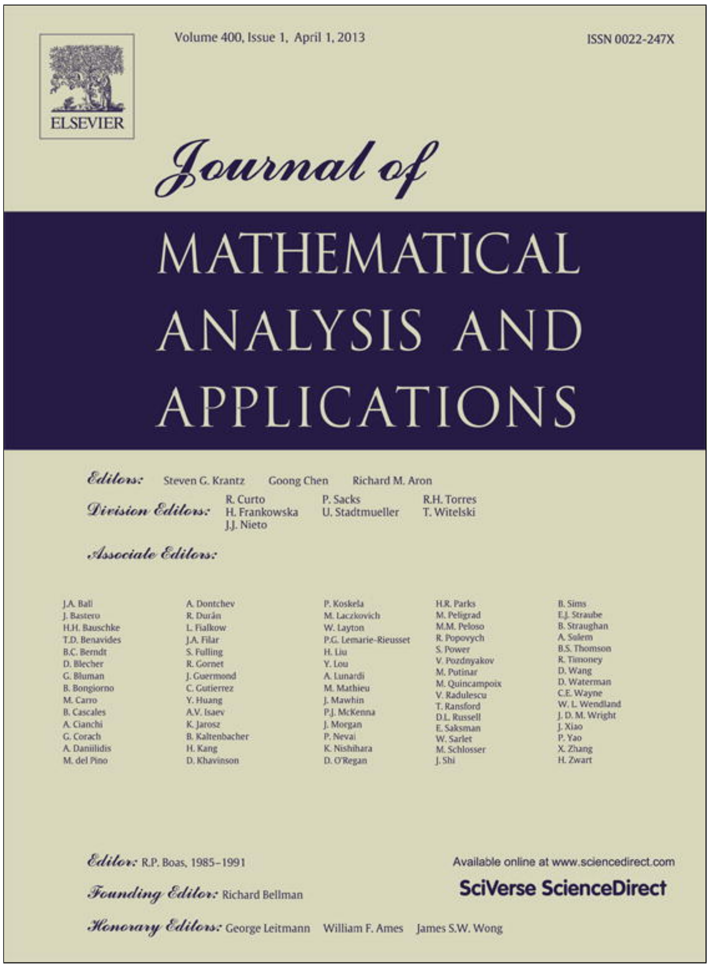
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Almost orthogonally additive functions☆

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ABSTRACT

If a function f, acting on a Euclidean space $\mathbb{R}^n$, is "almost" orthogonally additive in the sense that $f(x+y)=f(x)+f(y)$ for all $(x,y)\in\perp\setminus Z$, where Z is a "negligible" subset of the $(2n-1)$-dimensional manifold $\perp\subset\mathbb{R}^{2n}$, then f coincides almost everywhere with some orthogonally additive mapping.

1. Introduction

Let $(E,\langle\cdot|\cdot\rangle)$ be a real inner product space, $\dim E\geq 2$, and let $(G,+)$ be an Abelian group. A function $f\colon E\to G$ is called *orthogonally additive* iff it satisfies the equation

$$f(x+y)=f(x)+f(y) \tag{1}$$

for all $(x,y)\in\perp:=\{(x,y)\in E^2:\langle x|y\rangle=0\}$. It was proved independently by R. Ger, Gy. Szabó and J. Rätz [1, Corollary 10] that such a function has the form

$$f(x)=a\left(\|x\|^2\right)+b(x) \tag{2}$$

with some additive mappings $a\colon\mathbb{R}\to G$, $b\colon E\to G$ provided that G is uniquely 2-divisible. This divisibility assumption was dropped by K. Baron and J. Rätz [2, Theorem 1].

We are going to deal with the situation where equality (1) holds true for all orthogonal pairs (x,y) outside from a "negligible" subset of $\perp$. Considerations of this type go back to a problem [3], posed by P. Erdős, concerning the unconditional version of Cauchy's functional equation (1). It was solved by N. G. de Bruijn [4] and, independently, by W. B. Jurkat [5], and also generalized by R. Ger [6]. Similar research concerning mappings which preserve inner product was made by J. Chmieliński and J. Rätz [7] and by J. Chmieliński and R. Ger [8].

While studying unconditional functional equations, "negligible" sets are usually understood as the members of some proper linearly invariant ideal. Moreover, any such ideal of subsets of an underlying space X automatically generates another such ideal of subsets of X^2 via the Fubini theorem (see R. Ger [9] and M. Kuczma [10, Section 17.5]). However, we shall assume that equation (1) is valid for $(x,y)\in\perp\setminus Z$, where Z is "negligible" in $\perp$ (not only in E^2), and therefore the structure of $\perp$ should be appropriate to work with "linear invariance" and Fubini-type theorems. This is the reason why we restrict our attention to Euclidean spaces $\mathbb{R}^n$ and regard $\perp$ as a smooth $(2n-1)$-dimensional manifold lying in $\mathbb{R}^{2n}$.

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2. Preliminary results

For completeness let us recall some definitions concerning the manifold theory (for further information see, e.g., R. Abraham, J. E. Marsden and T. Ratiu [11], and L. W. Tu [12]). Let S be a topological space; by an *m-dimensional $\mathcal{C}^\infty$-atlas* we mean a family $\mathcal{A} = \{(U_i, \varphi_i)\}_{i\in I}$ such that $\{U_i\}_{i\in I}$ is an open covering of S, for each $i \in I$ the mapping φ_i is a homeomorphism which maps U_i onto an open subset of $\mathbb{R}^m$, and for each $i, j \in I$ the mapping $\varphi_i \circ \varphi_j^{-1}$ is a $\mathcal{C}^\infty$-diffeomorphism defined on $\varphi_j(U_i \cap U_j)$. Brouwer's theorem of dimension invariance implies that each two atlases on S are of the same dimension.

We say that atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ are *equivalent* iff $\mathcal{A}_1 \cup \mathcal{A}_2$ is an atlas. A *$\mathcal{C}^\infty$-differentiable structure* $\mathcal{D}$ on S is an equivalence class of atlases on S; the union $\bigcup \mathcal{D}$ forms a maximal atlas on S and any of its elements is called an *admissible chart*. By a *$\mathcal{C}^\infty$-differentiable manifold* (briefly: *manifold*) M we mean a pair $(S, \mathcal{D})$ of a topological space S and a $\mathcal{C}^\infty$-differentiable structure $\mathcal{D}$ on S; we shall then identify M with the space S for convenience. A manifold is called an *m-manifold* iff its every atlas is m-dimensional.

Having an m_1-manifold $M_1 = (S_1, \mathcal{D}_1)$ and an m_2-manifold $M_2 = (S_2, \mathcal{D}_2)$ we may define the *product manifold* $M_1 \times M_2 = (S_1 \times S_2, \mathcal{D}_1 \times \mathcal{D}_2)$, where the differentiable structure $\mathcal{D}_1 \times \mathcal{D}_2$ is generated by the atlas

$$\left\{(U_1 \times U_2, \varphi_1 \times \varphi_2) : (U_i, \varphi_i) \in \bigcup \mathcal{D}_i \text{ for } i = 1, 2\right\}.$$

Then $M_1 \times M_2$ forms an $(m_1 + m_2)$-manifold. For an arbitrary set $A \subset M_1 \times M_2$ and any point $x \in M_1$ we use the notation $A[x] = \{y \in M_2 : (x, y) \in A\}$.

In what follows, we consider only manifolds $M \subset \mathbb{R}^n$, for some $n \in \mathbb{N}$, equipped with the natural topology and a differentiable structure which is determined by the following condition: for every $x \in M$ there is a $\mathcal{C}^\infty$-diffeomorphism φ defined on an open set $U \subset \mathbb{R}^n$ with $x \in U$ such that $\varphi(M \cap U) = \varphi(U) \cap (\mathbb{R}^m \times \{0\})$, where m is the dimension of M. In particular, every open subset of $\mathbb{R}^n$ yields an n-manifold with the atlas consisting of a single identity map. Any set $M \subset \mathbb{R}^n$ satisfying the above condition forms a *submanifold* of $\mathbb{R}^n$ in the sense of [11, Definition 3.2.1], or a *regular submanifold* of $\mathbb{R}^n$ in the sense of [12, Definition 9.1]. Generally, if M_1 is an m_1-manifold and M_2 is an m_2-manifold, then M_1 is called a (*regular*) *submanifold* of M_2 iff $M_1 \subset M_2$ and for every $x \in M_1$ there is an admissible chart (U, φ) of M_2 with $x \in U$ such that $\varphi(M_1 \cap U) = \varphi(U) \cap (\mathbb{R}^{m_1} \times \{0\})$.

If M_1 and M_2 are manifolds with atlases $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, then a mapping $\Phi: M_1 \to M_2$ is said to be *of the class* $\mathcal{C}^\infty$ iff it is continuous and for all $(U, \varphi) \in \mathcal{A}_1$, $(V, \psi) \in \mathcal{A}_2$ the composition $\psi \circ \Phi \circ \varphi^{-1}$ is of the class $\mathcal{C}^\infty$ (in the usual sense) in its domain. This condition is independent of the choice of particular atlases generating differentiable structures of M_1 and M_2; see [11, Proposition 3.2.6]. We say that Φ is a *$\mathcal{C}^\infty$-diffeomorphism* iff Φ is a bijection between M_1 and M_2, and both Φ and Φ^{-1} are of the class $\mathcal{C}^\infty$. According to the above explanation, such a definition is compatible with the usual notion of a $\mathcal{C}^\infty$-diffeomorphism. If any $\mathcal{C}^\infty$-diffeomorphism between M_1 and M_2 exists, then we write $M_1 \sim M_2$. Of course, in such a case the manifolds M_1 and M_2 are of the same dimension.

Finally, a mapping $\Phi: M_1 \to M_2$ between an m_1-manifold M_1 and an m_2-manifold M_2 is called a *$\mathcal{C}^\infty$-immersion* [*$\mathcal{C}^\infty$-submersion*] iff it is of the class $\mathcal{C}^\infty$ and for every $x \in M_1$ there exist admissible charts (U, φ) and (V, ψ) of M_1 and M_2, respectively, such that $x \in U$, $\Phi(x) \in V$, and the derivative of the function $\psi \circ \Phi \circ \varphi^{-1}$ at any point of $\varphi(U)$ is an injective [a surjective] linear mapping from $\mathbb{R}^{m_1}$ to $\mathbb{R}^{m_2}$ (see [12, Proposition 8.12] for another, equivalent definition). We will find the following lemma useful; for the proof see R. W. R. Darling [13, Section 5.5.1].

Lemma 1. *Let M_1 be a submanifold of an open set $U \subset \mathbb{R}^{n_1}$ and M_2 be a submanifold of an open set $V \subset \mathbb{R}^{n_2}$. If $\Phi: U \to V$ is a $\mathcal{C}^\infty$-immersion [$\mathcal{C}^\infty$-submersion] with $\Phi(M_1) \subset M_2$, then the restriction $\Phi|_{M_1}: M_1 \to M_2$ is a $\mathcal{C}^\infty$-immersion [$\mathcal{C}^\infty$-submersion].*

Recall that given a non-empty set X a family $\mathscr{I} \subset 2^X$ is said to be a *proper σ-ideal* iff the following conditions hold:

(i) $X \notin \mathscr{I}$;
(ii) if $A \in \mathscr{I}$ and $B \subset A$, then $B \in \mathscr{I}$;
(iii) if $A_k \in \mathscr{I}$ for $k \in \mathbb{N}$, then $\bigcup_{k=1}^\infty A_k \in \mathscr{I}$.

From now on we suppose that for each $m \in \mathbb{N}$ a family $\mathscr{I}_m$ forms a proper σ-ideal of subsets of $\mathbb{R}^m$ satisfying the following conditions:

(H_0) $\{0\} \in \mathscr{I}_1$;
(H_1) if φ is a $\mathcal{C}^\infty$-diffeomorphism defined on an open set $U \subset \mathbb{R}^m$ and $A \in \mathscr{I}_m$, then $\varphi(A \cap U) \in \mathscr{I}_m$;
(H_2) if $m, n \in \mathbb{N}$ and $A \in \mathscr{I}_{m+n}$, then $\{x \in \mathbb{R}^m : A[x] \notin \mathscr{I}_n\} \in \mathscr{I}_m$;
(H_3) if $m, n \in \mathbb{N}$ and $A \in \mathscr{I}_n$, then $\mathbb{R}^m \times A \in \mathscr{I}_{m+n}$.

Note that by condition (H_1), non-empty open subsets of $\mathbb{R}^m$ do not belong to $\mathscr{I}_m$, whereas (H_0) and (H_1) imply that any countable subset of $\mathbb{R}^m$ is in $\mathscr{I}_m$.

Remark 1. The conditions (H_0)–(H_3) are satisfied in the following cases:

(a) when $\mathscr{I}_m$ consists of all first category subsets of $\mathbb{R}^m$, for $m \in \mathbb{N}$ (in this case (H_2) follows from the Kuratowski–Ulam theorem);
(b) when $\mathscr{I}_m$ consists of all Lebesgue measure zero subsets of $\mathbb{R}^m$, for $m \in \mathbb{N}$ (in this case (H_2) is just the classical Fubini theorem).

More generally, let μ be any measure defined on all Borel subsets of $\mathbb{R}$ and satisfying both (H_0) and (H_1). Let also

$$\mu_m = \underbrace{\mu \otimes \cdots \otimes \mu}_{m}$$

be the mth product measure and $\widetilde{\mu}_m$ be the completion of μ_m, for $m \in \mathbb{N}$. Then (H_0)–(H_3) are also satisfied in the two following cases:

(c) when $\mathscr{I}_m$ consists of all Borel subsets A of $\mathbb{R}^m$ with $\mu_m(A) = 0$ (condition (H_1) follows by induction from Fubini's theorem applied to the characteristic function of the Borel set $\varphi(A \cap U)$);
(d) when $\mathscr{I}_m$ consists of all μ_m-negligible subsets of $\mathbb{R}^m$, i.e., all $\widetilde{\mu}_m$-measurable sets $A \subset \mathbb{R}^m$ with $\widetilde{\mu}_m(A) = 0$ (if $A \in \mathscr{I}_m$ then A is contained in a Borel set having measure μ_m zero, thus condition (H_1) follows as in the preceding case).

For an arbitrary m-manifold $M \subset \mathbb{R}^n$ $(m \leq n)$ with an atlas $\mathcal{A} = \{(U_i, \varphi_i)\}_{i\in I}$ we define a proper σ-ideal $\mathscr{I}_M \subset 2^M$ by putting

$$\mathscr{I}_M = \{A \subset M : \varphi_i(A \cap U_i) \in \mathscr{I}_m \text{ for each } i \in I\}. \tag{3}$$

By condition (H_1), this definition does not depend on the particular choice of $\mathcal{A}$. Indeed, let $\{(V_j, \psi_j)\}_{j\in J}$ be another atlas of M, equivalent to $\mathcal{A}$. Fix any $A \in \mathscr{I}_M$ and $j \in J$. With the aid of Lindelöf's theorem we choose a countable set $I_0 \subset I$ such that $V_j \subset \bigcup_{i\in I_0} U_i$. For each $i \in I_0$ the mapping $\chi_i := \psi_j \circ \varphi_i^{-1}$ is a $\mathcal{C}^\infty$-diffeomorphism on $\varphi_i(V_j \cap U_i)$ and since $B_i := \varphi_i(A \cap V_j \cap U_i) \in \mathscr{I}_m$, we have $\psi_j(A \cap V_j \cap U_i) = \chi_i(B_i) \in \mathscr{I}_m$. Consequently, $\psi_j(A \cap V_j) = \bigcup_{i\in I_0} \psi_j(A \cap V_j \cap U_i) \in \mathscr{I}_m$. This shows that if $A \in \mathscr{I}_M$, then $\psi_j(A \cap V_j) \in \mathscr{I}_m$ for each $j \in J$. Analogously we obtain the reverse implication. Note that, by this definition, $\mathscr{I}_{\mathbb{R}^m} = \mathscr{I}_m$ for each $m \in \mathbb{N}$.

Lemma 2. *Let M_1 be an m_1-dimensional submanifold of an m_2-manifold $M_2 \subset \mathbb{R}^n$. Then*

(a) *$M_1 \in \mathscr{I}_{M_2}$, provided that $m_1 < m_2$;*
(b) *$\mathscr{I}_{M_1} \subset \mathscr{I}_{M_2}$.*

Proof. (a) By the submanifold property, we may choose an atlas $\mathcal{A}$ of M_2 such that $\varphi(M_1 \cap U) = \varphi(U) \cap (\mathbb{R}^{m_1} \times \{0\})$ for each $(U, \varphi) \in \mathcal{A}$. Since (H_0) and (H_3) imply $\mathbb{R}^{m_1} \times \{0\} \in \mathscr{I}_{m_2}$, we get $\varphi(M_1 \cap U) \in \mathscr{I}_{m_2}$, as desired.

(b) The case $m_1 < m_2$ reduces to assertion (a). If $m_1 = m_2$, then for every admissible chart of M_2 we have $\varphi(A \cap U) \in \mathscr{I}_{m_1} = \mathscr{I}_{m_2}$. $\square$

We can prove the following strengthening of condition (H_1).

Lemma 3. *If $\Phi\colon M_1 \to M_2$ is a $\mathcal{C}^\infty$-diffeomorphism between manifolds $M_1 \subset \mathbb{R}^{n_1}$, $M_2 \subset \mathbb{R}^{n_2}$, then for every $A \in \mathscr{I}_{M_1}$ we have $\Phi(A) \in \mathscr{I}_{M_2}$.*

Proof. Let $\mathcal{A}_1 = \{(U_i, \varphi_i)\}_{i\in I}$ and $\mathcal{A}_2 = \{(V_j, \psi_j)\}_{j\in J}$ be atlases generating the differentiable structures of M_1 and M_2, respectively. Let also m be the dimension of M_1 and M_2. Fix $j \in J$; we are to prove that $\psi_j(\Phi(A) \cap V_j) \in \mathscr{I}_m$. Choose a countable set $I_0 \subset I$ with $A \subset \bigcup_{i\in I_0} U_i$ and for each $i \in I_0$ define a $\mathcal{C}^\infty$-diffeomorphism $\chi_i = \psi_j \circ \Phi \circ \varphi_i^{-1}$. Then

$$\psi_j(\Phi(A) \cap V_j) \subset \bigcup_{i\in I_0} \chi_i(\varphi_i(A \cap U_i) \cap \mathrm{Dom}(\chi_i)), \tag{4}$$

where $\mathrm{Dom}(\chi_i)$ stands for the domain of χ_i. Moreover, since $A \in \mathscr{I}_{M_1}$, we have $\varphi_i(A \cap U_i) \in \mathscr{I}_m$ thus (H_1) implies that the both sets in (4) belong to $\mathscr{I}_m$. $\square$

Conditions (H_1), (H_2) imply a general version of Fubini's theorem.

Lemma 4. *Let $M_1 \subset \mathbb{R}^{n_1}$, $M_2 \subset \mathbb{R}^{n_2}$ be manifolds. If $A \in \mathscr{I}_{M_1\times M_2}$, then*

$$\{x \in M_1 : A[x] \notin \mathscr{I}_{M_2}\} \in \mathscr{I}_{M_1}.$$

Proof. Let $\{(U_i, \varphi_i)\}_{i\in I}$ and $\{(V_j, \psi_j)\}_{j\in J}$ be arbitrary countable atlases generating the differentiable structures of M_1 and M_2, respectively. Since $A \in \mathscr{I}_{M_1\times M_2}$, for each $i \in I$, $j \in J$ we have

$$B_{ij} := (\varphi_i \times \psi_j)(A \cap (U_i \times V_j)) \in \mathscr{I}_{m_1+m_2}.$$

Moreover,

$$B_{ij} = \{(\varphi_i(x), \psi_j(y)) \in \mathbb{R}^{m_1+m_2} : x \in U_i \text{ and } y \in A[x] \cap V_j\}$$

for $i \in I$, $j \in J$. Suppose, in search of a contradiction, that

$$Z := \{x \in M_1 : A[x] \notin \mathscr{I}_{M_2}\} \notin \mathscr{I}_{M_1}.$$

Then we may find $i_0 \in I$ with $Z \cap U_{i_0} \notin \mathscr{I}_{M_1}$. If for every $j \in J$ the set

$$C_j := \{x \in Z \cap U_{i_0} : A[x] \cap V_j \notin \mathscr{I}_{M_2}\}$$

belonged to $\mathscr{I}_{M_1}$, then we would have

$$Z \cap U_{i_0} = \{x \in Z \cap U_{i_0} : A[x] \notin \mathscr{I}_{M_2}\} = \bigcup_{j \in J} C_j \in \mathscr{I}_{M_1},$$

which is not the case. Therefore, we may find $j_0 \in J$ with $C_{j_0} \notin \mathscr{I}_{M_1}$. Define

$$B = \{(\varphi_{i_0}(x), \psi_{j_0}(y)) \in \mathbb{R}^{m_1+m_2} : x \in Z \cap U_{i_0} \text{ and } y \in A[x] \cap V_{j_0}\}$$

and note that $B \subset B_{i_0,j_0}$, whence $B \in \mathscr{I}_{m_1+m_2}$. However, $\varphi_{i_0}(C_{j_0}) \notin \mathscr{I}_{m_1}$ and for each $x \in C_{j_0}$ and $t = \varphi_{i_0}(x)$ we have

$$B[t] = \psi_{j_0}(A[x] \cap V_{j_0}) \notin \mathscr{I}_{m_2}.$$

This yields a contradiction with (H_2). □

Lemma 5. *If $\Phi: M_1 \to M_2$ is a $\mathcal{C}^\infty$-submersion between manifolds $M_1 \subset \mathbb{R}^{n_1}$, $M_2 \subset \mathbb{R}^{n_2}$, then for every $A \subset M_1$, $A \notin \mathscr{I}_{M_1}$ we have $\Phi(A) \notin \mathscr{I}_{M_2}$.*

Proof. By Lindelöf's theorem, there is a point $x_0 \in M_1$ such that for every its neighborhood $U \subset M_1$ we have $A \cap U \notin \mathscr{I}_{M_1}$. By the assumption, we may find admissible charts (U, φ) and (V, ψ) of M_1 and M_2, respectively, such that $x_0 \in U$, $\Phi(x_0) \in V$, $\varphi(A \cap U) \notin \mathscr{I}_{m_1}$ and the derivative of $\psi \circ \Phi \circ \varphi^{-1}$ at any point of $\varphi(U)$ is a surjection from $\mathbb{R}^{m_1}$ onto $\mathbb{R}^{m_2}$ (m_1, m_2 being the dimensions of M_1, M_2, respectively). Hence, obviously, $m_1 \geq m_2$ and there is a sequence $1 \leq i_1 < \cdots < i_{m_2} \leq m_1$ such that

$$\frac{\partial(\psi \circ \Phi \circ \varphi^{-1})}{\partial y_{i_1} \ldots \partial y_{i_{m_2}}}(\varphi(x_0)) \neq 0.$$

By decreasing the neighborhood U, we may guarantee that the above condition holds true for every $x \in U$ in the place of x_0, and that the mapping $\psi \circ \Phi \circ \varphi^{-1}$ is defined on the whole $\varphi(U)$. Let $\psi \circ \Phi \circ \varphi^{-1} = (G_1, \ldots, G_{m_2})$ and define a function $F = (F_1, \ldots, F_{m_1}): \varphi(U) \to \mathbb{R}^{m_1}$ by the formula

$$F_k(y) = \begin{cases} G_j(y) & \text{if } k = i_j \text{ for some } j \in \{1, \ldots, m_2\}, \\ y_k & \text{otherwise.} \end{cases}$$

Then for each $y \in \varphi(U)$ we have

$$\left| \frac{\partial F}{\partial y_1 \ldots \partial y_{m_1}}(y) \right| = \left| \frac{\partial(\psi \circ \Phi \circ \varphi^{-1})}{\partial y_{i_1} \ldots \partial y_{i_{m_2}}}(y) \right| \neq 0,$$

thus, decreasing U as required, we may assume that F is a $\mathcal{C}^\infty$-diffeomorphism. Enumerating the coordinates we may also modify F in such a way that it is still a $\mathcal{C}^\infty$-diffeomorphism and

$$F(\varphi(A \cap U)) \subset (\psi \circ \Phi \circ \varphi^{-1})(\varphi(A \cap U)) \times \mathbb{R}^{m_1 - m_2}. \tag{5}$$

In view of $\varphi(A \cap U) \notin \mathscr{I}_{m_1}$, condition (H_1) yields $F(\varphi(A \cap U)) \notin \mathscr{I}_{m_1}$, whence (5) and (H_3) imply $\psi(\Phi(A \cap U)) \notin \mathscr{I}_{m_2}$. Therefore, $\Phi(A \cap U) \notin \mathscr{I}_{M_2}$, since ψ is an admissible chart of M_2 defined on $\Phi(U)$. □

In a similar manner we obtain the next lemma.

Lemma 6. *If $\Phi: M_1 \to M_2$ is a $\mathcal{C}^\infty$-immersion between manifolds $M_1 \subset \mathbb{R}^{n_1}$, $M_2 \subset \mathbb{R}^{n_2}$, then for every $A \in \mathscr{I}_{M_1}$ we have $\Phi(A) \in \mathscr{I}_{M_2}$.*

From now on, let $n \geq 2$ be a fixed natural number and $\langle \cdot | \cdot \rangle$ be an arbitrary inner product in $\mathbb{R}^n$ inducing a norm which we denote by $\| \cdot \|$. For any set A we define $A^* = A \setminus \{0\}$, where the meaning of 0 is clear from the context. Let $\perp$ be the set of all pairs of orthogonal vectors from $\mathbb{R}^n$. Then $\perp^* = F^{-1}(0)$, where $F: (\mathbb{R}^n \times \mathbb{R}^n)^* \to \mathbb{R}$ is given by $F(x, y) = \langle x | y \rangle$. Since 0 is a regular value of F, it follows from [12, Theorem 9.11] that $\perp^*$ forms a $(2n-1)$-manifold (being also a regular submanifold of $(\mathbb{R}^n \times \mathbb{R}^n)^*$).

We may therefore make it precise what being "negligible" in $\perp$ means. Namely, we say that a set $Z \subset \perp$ has this property iff $Z \in \mathscr{I}_{\perp^*}$ and we will then write simply $Z \in \mathscr{I}_\perp$. We are now ready to formulate our main result which we shall prove in the last section. For notational convenience, if M is a manifold and some property, depending on a variable x, holds true for all $x \in M \setminus A$ with $A \in \mathscr{I}_M$, then we write that it holds $\mathscr{I}_M$-(a.e.).

Theorem. *Let $(G, +)$ be an Abelian group. If a function $f: \mathbb{R}^n \to G$ satisfies $f(x+y) = f(x) + f(y)$ $\mathscr{I}_\perp$-(a.e.), then there is a unique orthogonally additive function $g: \mathbb{R}^n \to G$ such that $f(x) = g(x)$ $\mathscr{I}_n$-(a.e.).*

Remark 2. According to Remark 1, the above theorem works whenever the ideal $\mathscr{I}_\perp$ is defined via formula (3) for $(\mathscr{I}_m)_{m=1}^\infty$ being one of the sequences of ideals described in (a)–(d).

In case (a) the ideal $\mathscr{I}_\perp$ consists of all first category subsets of $\perp^*$, regarded as a topological subspace of the Euclidean space $\mathbb{R}^{2n}$.

In case (b) the ideal $\mathscr{I}_\perp$ consists of all Lebesgue measure zero subsets of $\perp^*$. Recall that the Lebesgue measure on any regular submanifold M of $\mathbb{R}^n$ is defined with the aid of the formula

$$\mu_M(A) = \int_{\varphi(A)} |(\varphi^{-1})'(\boldsymbol{x})|\,\mathrm{d}\boldsymbol{x},$$

postulated for any admissible chart (U_φ, φ) of M and any set $A \subset M$ such that $A \subset U_\varphi$ and $\varphi(A) \subset \mathbb{R}^m$ is Lebesgue measurable.

Further examples are produced by the ideals $\mathscr{I}_m$ described in (c)–(d), in Remark 1. For instance, one may start with the α-dimensional Hausdorff measure $\mathscr{H}^\alpha$ (for some $0 < \alpha < 1$) defined on all Borel subsets (or on all Hausdorff measurable subsets) of $\mathbb{R}$ and, by using formula (3), induce a corresponding ideal $\mathscr{I}_\perp$. However, this ideal will not be the same as the ideal of all Borel (Hausdorff measurable) sets $A \subset \perp^*$ with $\mathscr{H}^{\alpha(2n-1)}(A) = 0$ (the $\alpha(2n-1)$-dimensional Hausdorff measure on the metric space $\perp^*$), since the product measure $\mathscr{H}^\alpha \otimes \mathscr{H}^\alpha$ need not be the Hausdorff measure $\mathscr{H}^{2\alpha}$ (consult [14, Section 3.1] and the references therein). This leads to the following question: Let $0 < \alpha < 1$. Is our Theorem true in the case where $\mathscr{I}_\perp$ is the set of all Borel (Hausdorff measurable) sets $A \subset \perp^*$ with $\mathscr{H}^{\alpha(2n-1)}(A) = 0$ and $\mathscr{I}_n$ is replaced by the ideal of all Borel (Hausdorff measurable) sets $B \subset \mathbb{R}^n$ with $\mathscr{H}^{\alpha n}(B) = 0$?

Before proceeding to further lemmas, let us note some preparatory observations. For any $x \in \mathbb{R}^n$ define

$$P_x = \{y \in \mathbb{R}^n : (x, y) \in \perp\},$$

which obviously forms an $(n-1)$-manifold diffeomorphic to $\mathbb{R}^{n-1}$, provided $x \neq 0$. We will need to "smoothly" identify the hyperplanes P_x, for different x's, with one "universal" space $\mathbb{R}^{n-1}$. By virtue of the Hairy Sphere Theorem, it is impossible to do for all $x \in (\mathbb{R}^n)^*$ in the case where n is odd. Nevertheless, it is an easy task when considering only the set of vectors for which one fixed coordinate is non-zero, e.g. the set

$$X := \mathbb{R}^{n-1} \times \mathbb{R}^*.$$

Namely, for an arbitrary $x \in X$ the vectors $x, e_1, \ldots, e_{n-1}$ are linearly independent, where e_i stands for the ith vector from the canonical basis of $\mathbb{R}^n$. Let $\mathscr{B}(x) = (y_i(x))_{i=0}^{n-1}$ be an orthonormal basis of $\mathbb{R}^n$ with $y_0(x) = x/\|x\|$, produced by the Gram–Schmidt process applied to the sequence $(x, e_1, \ldots, e_{n-1})$. Define $\psi_x\colon \mathbb{R}^n \to \mathbb{R}^n$ to be the mapping which to every $z \in \mathbb{R}^n$ assigns its coordinates with respect to $\mathscr{B}(x)$, i.e. $\psi_x(z) = \boldsymbol{Y}(x)^{-1}z$, where

$$\boldsymbol{Y}(x) = \left(\frac{x}{\|x\|}, y_1(x), \ldots, y_{n-1}(x)\right)$$

is the matrix formed from the column vectors. Define also $\Phi\colon X \times \mathbb{R}^n \to X \times \mathbb{R}^n$ by $\Phi(x, z) = (x, \psi_x(z))$. Plainly, Φ is a $\mathcal{C}^\infty$-mapping and its inverse $\Phi^{-1}(x, y) = (x, Y(x)y)$ is $\mathcal{C}^\infty$ as well. Therefore, Φ is a $\mathcal{C}^\infty$-diffeomorphism. Moreover, by the definition of ψ_x, the restriction $\psi_x|_{P_x}$ maps P_x onto $\{0\} \times \mathbb{R}^{n-1}$; hence we have

$$\Phi^{-1}\left(X \times (\{0\} \times \mathbb{R}^{n-1})\right) = \{(x, z) \in \perp^* : x \in X\} =: \perp'. \tag{6}$$

Making use of [12, Theorem 11.20] and an easy fact that the restriction of a $\mathcal{C}^\infty$ mapping to a submanifold of its domain is $\mathcal{C}^\infty$ again,[1] we infer by (6) that $\Phi|_{\perp'}$ yields a $\mathcal{C}^\infty$-diffeomorphism between $\perp'$ and $X \times (\{0\} \times \mathbb{R}^{n-1})$.

Consequently, if a function $h\colon \mathbb{R}^n \to G$ satisfies $h(x+y) = h(x) + h(y)\,\mathscr{I}_\perp$-(a.e.), then with the notation

$$Z(h) := \{(x, y) \in \perp^* : h(x+y) \neq h(x) + h(y)\}$$

it follows from Lemmas 3 and 4 that

$$\left\{x \in X : \{\psi_x(z) : (x, z) \in Z(h)\} \notin \mathscr{I}_{\{0\}\times\mathbb{R}^{n-1}}\right\} \in \mathscr{I}_X.$$

Since $P_x \sim \{0\} \times \mathbb{R}^{n-1}$, by the mapping $\psi_x|_{P_x}$ for $x \in X$, we infer that the set

$$D(h) := \{x \in X : h(x+y) = h(x) + h(y)\ \mathscr{I}_{P_x}\text{-(a.e.)}\}$$

satisfies $X \setminus D(h) \in \mathscr{I}_X$. For any $x \in \mathbb{R}^n$ put

$$E_x(h) = \{y \in P_x : h(x+y) = h(x) + h(y)\};$$

then $P_x \setminus E_x(h) \in \mathscr{I}_{P_x}$, provided $x \in D(h)$.

We end this section with a lemma, which will be useful in the "odd" part of the proof of our theorem. Despite it will be applied only in the case $n = 2$, we present it in full generality, since the lemma seems to be interesting independently on

1 In the sequel, we will be using these two assertions without explicit mentioning.

the problem considered. Let S^{n-1} be the unit sphere of the normed space $(\mathbb{R}^n, \|\cdot\|)$. Since the function $F\colon \mathbb{R}^n \to \mathbb{R}$ given by $F(x) = \|x\|^2$ is $\mathcal{C}^\infty$ with the regular value 1 and $S^{n-1} = F^{-1}(1)$, we infer that S^{n-1} is an $(n-1)$-manifold.

Lemma 7. *If $A \in \mathscr{I}_{S^{n-1}}$, then there exists an orthogonal basis $(x_1, \dots, x_n)$ of $\mathbb{R}^n$ such that $x_i \in S^{n-1} \setminus A$ for each $i \in \{1, \dots, n\}$.*

Proof. It is enough to prove the assertion in the case where $\langle\cdot|\cdot\rangle$ is the standard inner product in $\mathbb{R}^n$, since between any two inner product structures in $\mathbb{R}^n$ there is a linear isometry, which yields a $\mathcal{C}^\infty$-diffeomorphism between their unit spheres.

Consider the group $\mathrm{GL}(n)$ of $n \times n$ real matrices with non-zero determinant. It may be identified with an open subset of $\mathbb{R}^{n^2}$ and hence—it is an n^2-manifold. It is well-known that the orthogonal group

$$\mathrm{O}(n) = \{\boldsymbol{A} \in \mathrm{GL}(n) : \boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{I}_n\}$$

forms a submanifold of $\mathrm{GL}(n)$ and its dimension equals $n(n-1)/2$ (see [11, Section 3.5.5C]). For any $i \in \{1, \dots, n\}$ let $\pi_i\colon \mathrm{O}(n) \to S^{n-1}$ be given by $\pi_i(\boldsymbol{A}) = \boldsymbol{A}e_i$ (which is nothing else but the ith column vector of $\boldsymbol{A}$). Then π_i is the restriction of the mapping $\overline{\pi}_i\colon \mathrm{GL}(n) \to \mathbb{R}^n$ defined by the formula analogous to the previous one. Since

$$\mathrm{D}\overline{\pi}_i(\boldsymbol{A})\boldsymbol{B} = \boldsymbol{B}e_i \quad \text{for } \boldsymbol{A} \in \mathrm{GL}(n),\ \boldsymbol{B} \in \mathbb{R}^{n^2},$$

the derivative $\mathrm{D}\overline{\pi}_i(\boldsymbol{A})$ is onto for any $\boldsymbol{A} \in \mathrm{GL}(n)$, thus $\overline{\pi}_i$ is a $\mathcal{C}^\infty$-submersion. By Lemma 1, π_i is a $\mathcal{C}^\infty$-submersion as well.

Now, suppose on the contrary that each orthonormal basis of $\mathbb{R}^n$ has at least one entry belonging to A. In other words, for each $\boldsymbol{A} \in \mathrm{O}(n)$ there is $i \in \{1, \dots, n\}$ with $\pi_i(\boldsymbol{A}) \in A$, i.e.

$$\mathrm{O}(n) = \bigcup_{i=1}^{n} \pi_i^{-1}(A).$$

Therefore, for a certain $i \in \{1, \dots, n\}$ we would have $\pi_i^{-1}(A) \notin \mathscr{I}_{\mathrm{O}(n)}$. However, $A = \pi_i(\pi_i^{-1}(A)) \in \mathscr{I}_{S^{n-1}}$, which contradicts the assertion of Lemma 5, as π_i is a $\mathcal{C}^\infty$-submersion. $\square$

3. Proof of the theorem

For the uniqueness part of our Theorem suppose that there are two orthogonally additive functions g_1 and g_2 equal to f $\mathscr{I}_n$-(a.e.). By the general form (2) of orthogonally additive mappings, we see that both g_1 and g_2 satisfy the Fréchet functional equation $\Delta_y^3 g(x) = 0$; thus arguing as in the proof of the uniqueness part of [15, Theorem 1], or making use of [10, Lemma 17.7.1], we get $g_1 = g_2$.

The proof of existence relies on some ideas from [2,1]. Assume G and f are as in the theorem. We start with the following trivial observation.

Lemma 8. *The functions $f_1, f_2\colon \mathbb{R}^n \to G$ given by*

$$f_1(x) = f(x) - f(-x) \quad \textit{and} \quad f_2(x) = f(x) + f(-x)$$

satisfy

$$f_1(x+y) = f_1(x) + f_1(y) \quad \textit{and} \quad f_2(x+y) = f_2(x) + f_2(y) \quad \mathscr{I}_\perp\text{-(a.e.)}.$$

In the sequel we will be using hypothesis (H_0)–(H_3) and Lemmas 2–4 without explicit mentioning.

For $k, m \in \mathbb{N}$ with $2 \le k \le m$ we define $\mathrm{O}(k, m)$ as the set of all k-tuples of mutually orthogonal (with respect to the usual scalar product) vectors from $\mathbb{R}^m$ with at most one of them being zero. Put

$$\mathcal{R}_{k,m} = \{(x^{(1)}, \dots, x^{(k)}) \in (\mathbb{R}^m)^k : x^{(i)} = 0 \text{ for at most one } i = 1, \dots, k\}.$$

Then $\mathrm{O}(k.m) = F^{-1}(0)$, where $F\colon \mathcal{R}_{k,m} \to \mathbb{R}^{\frac{k(k-1)}{2}}$ is given by

$$\begin{aligned} F(x^{(1)}, \dots, x^{(k)}) = (&\langle x^{(1)}|x^{(2)}\rangle, \langle x^{(1)}|x^{(3)}\rangle, \dots, \langle x^{(1)}|x^{(k)}\rangle, \\ &\langle x^{(2)}|x^{(3)}\rangle, \dots, \langle x^{(2)}|x^{(k)}\rangle, \\ &\vdots \\ &\langle x^{(k-1)}|x^{(k)}\rangle). \end{aligned}$$

Since 0 is a regular value of F, [12, Theorem 9.11] implies that $\mathrm{O}(k, m)$ is a submanifold of $\mathbb{R}^{km}$ with dimension $km - \frac{1}{2}k(k-1)$. In particular, $\mathrm{O}(2, n) = \perp^*$.

Lemma 9. *Let $k \in \mathbb{N}$, $k \ge 2$ and let $A \subset \mathrm{O}(2, k)$ be a set such that*

$$\{(x^{(1)}, \dots, x^{(k)}) \in \mathrm{O}(k, k) : (x^{(1)}, x^{(2)}) \in A\} \in \mathscr{I}_{\mathrm{O}(k,k)}.$$

Then $A \in \mathscr{I}_{\mathrm{O}(2,k)}$.

Proof. Denote the above subset of $\mathrm{O}(k,k)$ by B. We may clearly assume that for each $(x^{(1)},x^{(2)})\in A$ we have $x^{(1)}\neq 0\neq x^{(2)}$. For $i,j\in\{1,\dots,k\}$ define

$$D_{ij}=\left\{(x^{(1)},x^{(2)})\in \mathrm{O}(2,k):\det\begin{pmatrix}x_i^{(1)} & x_j^{(1)}\\ x_i^{(2)} & x_j^{(2)}\end{pmatrix}\neq 0\right\},$$
$$B_{ij}=\{(x^{(1)},\dots,x^{(k)})\in B:(x^{(1)},x^{(2)})\in D_{ij}\}.$$

We will show that

$$A=\bigcup_{\substack{i,j=1\\ i\neq j}}^{k}(A\cap D_{ij})\quad\text{and}\quad B=\bigcup_{\substack{i,j=1\\ i\neq j}}^{k}B_{ij}. \tag{7}$$

For the former equality suppose that for some $(x^{(1)},x^{(2)})\in A$ and each pair of indices $1\le i,j\le k$, $i\neq j$, we have

$$\det\begin{pmatrix}x_i^{(1)} & x_j^{(1)}\\ x_i^{(2)} & x_j^{(2)}\end{pmatrix}=0. \tag{8}$$

Then for each $1\le i\le k$ we have $x_i^{(1)}=0$ if and only if $x_i^{(2)}=0$. Indeed, choosing any $1\le j\le k$ such that $x_j^{(1)}\neq 0$ we see from (8) that $x_i^{(1)}=0$ implies $x_i^{(2)}=0$; the reverse implication holds by symmetry. Now, let $1\le i_1<\dots<i_\ell\le k$ be the indices of all non-zero coordinates of $x^{(1)}$ (and $x^{(2)}$). For each pair of $1\le i,j\le k$ one of the rows of the determinant in (8) is a multiple of the other. Applying this observation consecutively for the pairs $(i_1,i_2),(i_2,i_3),\dots,(i_{\ell-1},i_\ell)$ we infer that $x^{(1)}$ and $x^{(2)}$ are parallel. Since they are also orthogonal, one of them should be zero which is the case we have excluded. The former equality in (7) is thus proved, and its easy consequence is the latter one.

We are now to show that $A\cap D_{ij}\in\mathscr{I}_{\mathrm{O}(2,k)}$ for each pair of indices $i,j\in\{1,\dots,k\}$ with $i\neq j$. So, fix any such pair and assume that $i<j$. Then for every $(x^{(1)},x^{(2)})\in D_{ij}$ the vectors:

$$x^{(1)},x^{(2)},e_1,\dots,e_{i-1},e_{i+1},\dots,e_{j-1},e_{j+1},\dots,e_k$$

form a basis of $\mathbb{R}^k$. Let

$$\mathscr{B}(x^{(1)},x^{(2)})=\big(y_i(x^{(1)},x^{(2)})\big)_{i=1}^{k}$$

be an orthonormal basis produced by the Gram–Schmidt process applied to that sequence of vectors. Since $x^{(1)}$ and $x^{(2)}$ are orthogonal, we have

$$y_1(x^{(1)},x^{(2)})=\frac{x^{(1)}}{\|x^{(1)}\|}\quad\text{and}\quad y_2(x^{(1)},x^{(2)})=\frac{x^{(2)}}{\|x^{(2)}\|}.$$

For $(x^{(1)},x^{(2)})\in D_{ij}$ define $\vartheta_{x^{(1)},x^{(2)}}\colon\mathbb{R}^k\to\mathbb{R}^k$ as the mapping which to every $z\in\mathbb{R}^k$ assigns its coordinates with respect to $\mathscr{B}(x^{(1)},x^{(2)})$, i.e.

$$\vartheta_{x^{(1)},x^{(2)}}(z)=\boldsymbol{Y}(x^{(1)},x^{(2)})^{-1}z,$$

where

$$\boldsymbol{Y}(x^{(1)},x^{(2)})=\left(\frac{x^{(1)}}{\|x^{(1)}\|},\frac{x^{(2)}}{\|x^{(2)}\|},y_3(x^{(1)},x^{(2)}),\dots,y_k(x^{(1)},x^{(2)})\right)$$

is formed from the column vectors. Obviously, every z belonging to the orthogonal complement $V(x^{(1)},x^{(2)})^\perp$ of the subspace spanned by $x^{(1)}$ and $x^{(2)}$ is mapped onto a certain vector of the form $(0,0,t_3,\dots,t_k)$ which may be naturally identified with an element of $\mathbb{R}^{k-2}$. Hence, we get a linear isomorphism $\gamma_{x^{(1)},x^{(2)}}\colon V(x^{(1)},x^{(2)})^\perp\to\mathbb{R}^{k-2}$ and we may define a mapping

$$\Gamma\colon\{(x^{(1)},\dots,x^{(k)})\in\mathrm{O}(k,k):(x^{(1)},x^{(2)})\in D_{ij}\}\to\big(\mathrm{O}(2,k)\cap D_{ij}\big)\times\mathrm{O}(k-2,k-2)$$

by the formula

$$\Gamma(x^{(1)},\dots,x^{(k)})=\big((x^{(1)},x^{(2)}),(\gamma_{x^{(1)},x^{(2)}}(x^{(3)}),\dots,\gamma_{x^{(1)},x^{(2)}}(x^{(k)}))\big).$$

The definition is well-posed, since $\vartheta_{x^{(1)},x^{(2)}}$, and hence also $\gamma_{x^{(1)},x^{(2)}}$, is an isometry for each $(x^{(1)},x^{(2)})\in D_{ij}$. Moreover, it is easily seen that Γ is a $\mathcal{C}^\infty$-diffeomorphism (the formulas of the Gram–Schmidt procedure are $\mathcal{C}^\infty$).

It easily follows from $B\in\mathscr{I}_{\mathrm{O}(k,k)}$ that B_{ij} belongs to the corresponding ideal of subsets of

$$\{(x^{(1)},\dots,x^{(k)})\in\mathrm{O}(k,k):(x^{(1)},x^{(2)})\in D_{ij}\};$$

thus $\Gamma(B_{ij})$ belongs to the ideal corresponding to $(\mathrm{O}(2,k)\cap D_{ij})\times \mathrm{O}(k-2,k-2)$. Finally, observe that

$$\Gamma(B_{ij}) = (A\cap D_{ij})\times \mathrm{O}(k-2,k-2),$$

which yields $A\cap D_{ij}\in \mathscr{I}_{\mathrm{O}(2,k)\cap D_{ij}}$ and hence also $A\cap D_{ij}\in \mathscr{I}_{\mathrm{O}(2,k)}$. $\square$

Lemma 10. *If an odd function $h\colon \mathbb{R}^n\to G$ satisfies $h(x+y)=h(x)+h(y)$ $\mathscr{I}_\perp$-(a.e.), then there is an additive function $b\colon \mathbb{R}^n\to G$ such that $h(x)=b(x)$ $\mathscr{I}_n$-(a.e.).*

Proof. Due to some isometry formalities, we may suppose $\langle\cdot|\cdot\rangle$ to be the standard inner product in $\mathbb{R}^n$.

Define

$$W=\{x=(x_1,\ldots,x_n)\in\mathbb{R}^n : x_i=0 \text{ for some } i\}$$

and

$$S_+^{n-1}=\{x=(x_1,\ldots,x_n)\in S^{n-1} : x_n>0\}.$$

Since S_+^{n-1} is an open subset of S^{n-1}, it is an $(n-1)$-manifold. For any $x\in S_+^{n-1}$ let

$$T_x=\{(\lambda,y)\in\mathbb{R}^*\times P_x^* : \lambda^2=\|y\|^2\}.$$

Define a map $\overline{\Phi}_x\colon \mathbb{R}^*\times P_x^*\to\mathbb{R}^n\times\mathbb{R}^n$ by

$$\overline{\Phi}_x(\lambda,y)=\Big(\lambda x+y,\ \frac{\|y\|^2}{\lambda}x-y\Big), \tag{9}$$

and set $\Phi_x=\overline{\Phi}_x|_{(\mathbb{R}^*\times P_x^*)\setminus T_x}$. Let also $Q(x)=\Phi_x\big((\mathbb{R}^*\times P_x^*)\setminus T_x\big)\subset\perp^*$. We are going to show that for every $x\in P:=S_+^{n-1}\setminus W$ the set $Q(x)$ forms a submanifold of $\perp^*$.

At the moment, let $x\in S_+^{n-1}$. For brevity, denote $\mu=\mu(\lambda,y)=\|y\|^2/\lambda$. It is easily seen that for each $(t,u)=(\lambda x+y,\mu x-y)\in Q(x)$ all four vectors: t,u,x,y belong to the subspace $V(t,x)$ of $\mathbb{R}^n$ spanned by t and x. Choose an arbitrary non-zero vector $z(t,x)\in V(t,x)$, orthogonal to x. Then $z(t,x)$ is collinear with y; hence the equality $t=\lambda x+y$ represents t in terms of the basis $(x,z(t,x))$ of $V(t,x)$. Therefore, λ and y are uniquely determined by t, which proves that Φ_x is injective.

In order to show that Φ_x^{-1} is continuous fix an arbitrary $(t,u)\in Q(x)$. Now, put $z(t,x)=\langle t|x\rangle x-t$; then $(x,z(t,x))$ is an orthogonal basis of $V(t,x)$. Since $t=\lambda x+y$ for certain $\lambda\in\mathbb{R}^*$ and $y\in P_x^*$, we have $t=\lambda x+\alpha z(t,x)$ for some $\alpha\in\mathbb{R}$, whence we find that $\lambda=\langle t|x\rangle$ and $y=t-\langle t|x\rangle x$. We have thus shown that Φ_x is a homeomorphism.

Now, fix $x\in P$. We shall prove that Φ_x is a $\mathcal{C}^\infty$-immersion. To this end put

$$V_x=\Big\{(\lambda,y)\in\mathbb{R}^*\times(\mathbb{R}^n)^* : \lambda=\langle x|y\rangle\pm\sqrt{\langle x|y\rangle^2+\|y\|^2}\Big\}$$

and define a mapping $\hat{\Phi}_x\colon (\mathbb{R}^*\times(\mathbb{R}^n)^*)\setminus V_x\to\mathbb{R}^n\times\mathbb{R}^n$ by the formula analogous to (9). Then $(\mathbb{R}^*\times P_x^*)\setminus T_x$ is a submanifold of $(\mathbb{R}^*\times(\mathbb{R}^n)^*)\setminus V_x$. Let $(\lambda,y)\in(\mathbb{R}^*\times(\mathbb{R}^n)^*)\setminus V_x$. If we show that the derivative $\mathrm{D}\hat{\Phi}_x(\lambda,y)$ is injective, then, in view of Lemma 1, we will be done. Since

$$\mathrm{D}\hat{\Phi}_x(\lambda,y)=\left(\begin{array}{c|cccc} x_1 & 1 & 0 & \ldots & 0\\ x_2 & 0 & 1 & \ldots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ x_n & 0 & 0 & \ldots & 1\\ \hline \dfrac{\partial(\mu x-y)}{\partial\lambda} & & \multicolumn{2}{c}{\dfrac{\partial(\mu x-y)}{\partial y}} & \end{array}\right),$$

we immediately get that $\operatorname{rank}\mathrm{D}\hat{\Phi}_x(\lambda,y)\ge n$, where the equality occurs only if the first column vector is a linear combination of the remaining n column vectors with coefficients $x_1,\ldots,x_n$. However, this would imply that for each $i\in\{1,\ldots,n\}$ we have

$$\frac{\partial(\mu x_i-y_i)}{\partial\lambda}=\sum_{j=1}^n x_j\frac{\partial(\mu x_i-y_i)}{\partial y_j},$$

i.e.

$$\lambda^2-2\langle x|y\rangle\lambda-\|y\|^2=0,$$

which is not the case, since $(\lambda,y)\notin V_x$. As a result, we obtain $\operatorname{rank}\mathrm{D}\hat{\Phi}_x(\lambda,y)=n+1$; thus $\mathrm{D}\hat{\Phi}_x(\lambda,y)$ is injective.

We have shown that Φ_x is an embedding (i.e. homeomorphic $\mathcal{C}^\infty$-immersion) of $(\mathbb{R}^* \times P_x^*) \setminus T_x$ into $\perp^*$. By virtue of [12, Theorem 11.17], its image $Q(x)$ is a submanifold of $\perp^*$.

Observe that the manifolds $Q(x)$, for $x \in P$, are $\mathcal{C}^\infty$-diffeomorphic to each other. Indeed, by the remarks following the statement of our Theorem, for each $x \in X$ the function $\Psi_x : \mathbb{R}^* \times P_x^* \to \mathbb{R}^* \times (\mathbb{R}^{n-1})^*$ defined by the formula

$$\Psi_x(\lambda, y) = (\lambda, \widetilde{\psi}_x(y)), \tag{10}$$

where $\psi_x(y) = \boldsymbol{Y}(x)^{-1}y$ is defined as earlier and the tilde operator deletes the first coordinate (which equals 0 for $y \in P_x$), is a $\mathcal{C}^\infty$-diffeomorphism. Moreover, Ψ_x maps $(\mathbb{R}^* \times P_x^*) \setminus T_x$ onto the set

$$U := \{(\lambda, y) \in \mathbb{R}^* \times (\mathbb{R}^{n-1})^* : \lambda^2 \neq \|y\|^2\},$$

which follows from the fact that $\widetilde{\psi}_x$ is an isometry. Therefore, for each $x, y \in P$, the mapping $\Phi_y \circ \Psi_y^{-1} \circ \Psi_x \circ \Phi_x^{-1}$ yields a $\mathcal{C}^\infty$-diffeomorphism between $Q(x)$ and $Q(y)$. So, we pick any $x_0 \in P$ and we regard the set $Q := Q(x_0)$ as a "model" manifold for all $Q(x)$'s.

Define

$$\perp^{(1)} = \{(t, u) \in \perp^* : t_n + u_n \neq 0,\ t \neq 0,\ u \neq 0 \text{ and } \|t\| \neq \|u\|\}$$

(which is an open subset, and hence it is a submanifold, of $\perp^*$) and observe that

$$\perp^{(1)} = \bigcup_{x \in S_+^{n-1}} Q(x). \tag{11}$$

In fact, for any $(t, u) \in \perp^{(1)}$ put

$$x = \operatorname{sgn}(t_n + u_n) \frac{t + u}{\|t + u\|}. \tag{12}$$

Then $x \in S_+^{n-1}$ and $(t, u) \in Q(x)$. Indeed, if we choose any $y_0 \in P_x^* \cap V(t, u)$ with $\|y_0\| = 1$ (which is unique up to a sign), then t and u are represented in terms of the basis (x, y_0) of $V(t, u)$ as follows:

$$t = \langle t|x\rangle x + \langle t|y_0\rangle y_0 \quad \text{and} \quad u = \langle u|x\rangle x + \langle u|y_0\rangle y_0,$$

and we have

$$\langle t|y_0\rangle = \langle t + u|y_0\rangle - \langle u|y_0\rangle = \pm\|t + u\|\langle x|y_0\rangle - \langle u|y_0\rangle = -\langle u|y_0\rangle.$$

Hence, after substitution $\lambda = \langle t|x\rangle$ and $y = \langle t|y_0\rangle y_0$, we obtain $t = \lambda x + y$ and $u = \langle u|x\rangle x - y$. The coefficient $\langle u|x\rangle$ equals $\|y\|^2/\lambda$, since $\langle t|u\rangle = \langle x|y\rangle = 0$. Moreover, $\lambda \neq 0$, $y_0 \neq 0$, and it follows from $\|t\| \neq \|u\|$ that $\lambda^2 \neq \langle u|x\rangle^2 = \|y\|^4/\lambda^2$, which gives $\lambda^2 \neq \|y\|^2$. Consequently, $(t, u) \in Q(x)$ and thus we have proved the inclusion "$\subseteq$". The reverse inclusion is a straightforward calculation.

We shall now prove that the mapping $\Lambda : S_+^{n-1} \times U \to \perp^{(1)}$ defined by

$$\Lambda(x, \lambda, y) = \Phi_x \circ \Psi_x^{-1}(\lambda, y)$$

is a $\mathcal{C}^\infty$-diffeomorphism.

First, in view of (11), it is easily seen that the image of Λ is $\perp^{(1)}$. According to the definition, Λ is $\mathcal{C}^\infty$. Moreover, for each $(t, u) = \Phi_x\big(\lambda, \widetilde{\psi}_x^{-1}(y)\big) \in Q(x)$ we have

$$\left(\lambda + \frac{\|\widetilde{\psi}_x^{-1}(y)\|^2}{\lambda}\right) x = t + u, \tag{13}$$

which, jointly with the fact that $x \in S_+^{n-1}$, uniquely determines x. By the injectivity of Φ_x, we infer that λ and y are then uniquely determined by t and u as well. Therefore, Λ is injective.

In order to get a formula for Λ^{-1}, observe that for each $(t, u) = \Phi_x\big(\lambda, \widetilde{\psi}_x^{-1}(y)\big) \in \perp^{(1)}$ equality (13) yields (12). This means that x is expressed as a function of t and u, which is $\mathcal{C}^\infty$ on both components of the set $\perp^{(1)}$. By the formula for Φ_x^{-1}, we get

$$\lambda = \operatorname{sgn}(t_n + u_n) \frac{\langle t|t + u\rangle}{\|t + u\|} \quad \text{and} \quad y = \widetilde{\psi}_x\left(t - \frac{\langle t|t + u\rangle}{\|t + u\|^2}(t + u)\right),$$

and since the value of $\widetilde{\psi}_x$ at a given point is a $\mathcal{C}^\infty$ function of x, we infer that Λ^{-1} is $\mathcal{C}^\infty$. Consequently, Λ is a $\mathcal{C}^\infty$-diffeomorphism.

Let $\chi : \perp^{(1)} \to S_+^{n-1} \times Q$ be given by

$$\chi = (\mathrm{id}_{S_+^{n-1}} \times \overline{\Phi}_{x_0}) \circ (\mathrm{id}_{S_+^{n-1}} \times \Psi_{x_0}^{-1}) \circ \Lambda^{-1};$$

then χ is a $\mathcal{C}^\infty$-diffeomorphism. Since $Z(h) \in \mathscr{I}_\perp$ and $\perp^{(1)}$ is an open subset of $\perp^*$, we have $Z(h) \cap \perp^{(1)} \in \mathscr{I}_{\perp^{(1)}}$. Therefore,

$$\{x \in S_+^{n-1} : \chi(Z(h) \cap \perp^{(1)})[x] \notin \mathscr{I}_Q\} \in \mathscr{I}_{S_+^{n-1}}. \tag{14}$$

Let $x \in S_+^{n-1}$. For any $q \in Q$ we have

$$q \in \chi(Z(h) \cap \perp^{(1)})[x] \iff (x, q) \in \chi(Z(h) \cap \perp^{(1)}) \iff \chi^{-1}(x, q) \in Z(h) \cap \perp^{(1)}.$$

Plainly,

$$\chi^{-1} = \Lambda \circ (\mathrm{id}_{S_+^{n-1}} \times \Psi_{x_0}) \circ (\mathrm{id}_{S_+^{n-1}} \times \overline{\Phi}_{x_0})^{-1},$$

so the last condition is equivalent to $\Lambda(x, (\Psi_{x_0} \circ \overline{\Phi}_{x_0}^{-1})(q)) \in Z(h)$. We have thus shown that

$$\chi(Z(h) \cap \perp^{(1)})[x] = \{q \in Q : \Lambda(x, (\Psi_{x_0} \circ \overline{\Phi}_{x_0}^{-1})(q)) \in Z(h)\}.$$

Since the map $\Psi_{x_0} \circ \overline{\Phi}_{x_0}^{-1} : Q \to U$ is a diffeomorphism, axiom (H_1) and Lemma 3 imply that:

$$\begin{aligned}
\chi(Z(h) \cap \perp^{(1)})[x] \notin \mathscr{I}_Q &\iff \{(\lambda, y) \in U : \Lambda(x, \lambda, y) \in Z(h)\} \notin \mathscr{I}_n \\
&\iff \{(\lambda, y) \in U : \overline{\Phi}_x(\lambda, \widetilde{\psi}_x^{-1}(y)) \in Z(h)\} \notin \mathscr{I}_n \\
&\iff \{(\lambda, y) \in \mathbb{R}^* \times (\mathbb{R}^{n-1})^* : \overline{\Phi}_x(\lambda, \widetilde{\psi}_x^{-1}(y)) \in Z(h)\} \notin \mathscr{I}_n \\
&\iff \{(\lambda, y) \in \mathbb{R}^* \times P_x^* : \overline{\Phi}_x(\lambda, y) \in Z(h)\} \notin \mathscr{I}_{\mathbb{R}^* \times P_x^*} \\
&\iff \{(\lambda, y) \in (\mathbb{R}^* \times P_x^*) \setminus T_x : \Phi_x(\lambda, y) \in Z(h)\} \notin \mathscr{I}_{(\mathbb{R}^* \times P_x^*) \setminus T_x} \\
&\iff Z(h) \cap Q(x) \notin \mathscr{I}_{Q(x)}.
\end{aligned}$$

Thus (14) gives

$$\{x \in P : Z(h) \cap Q(x) \notin \mathscr{I}_{Q(x)}\} \in \mathscr{I}_{S_+^{n-1}}.$$

Since $S_+^{n-1} \setminus P \in \mathscr{I}_{S_+^{n-1}}$, we have also

$$Z(h) \cap Q(x) \in \mathscr{I}_{Q(x)} \quad \mathscr{I}_{S_+^{n-1}}\text{-(a.e.)}. \tag{15}$$

For any $x \in S_+^{n-1}$ define $\Gamma_x : \mathbb{R}^* \times P_x^* \to \perp^*$ and $\Theta_x : \mathbb{R}^* \times P_x^* \to \perp^*$ as

$$\Gamma_x(\lambda, y) = \left(\frac{\|y\|^2}{\lambda}x, -y\right) \quad \text{and} \quad \Theta_x(\lambda, y) = (\lambda x, y),$$

and put $R(x) = \Gamma_x(\mathbb{R}^* \times P_x^*)$, $S(x) = \Theta_x(\mathbb{R}^* \times P_x^*)$. An argument similar to the one above shows that $R(x)$, for $x \in S_+^{n-1}$, are submanifolds of $\perp^*$, $\mathcal{C}^\infty$-diffeomorphic to each other, and the same is true for $S(x)$'s. Moreover, the set

$$\perp^{(2)} := \{(t, u) \in \perp^* : t_n \neq 0 \text{ and } u \neq 0\} = \bigcup_{x \in S_+^{n-1}} R(x) = \bigcup_{x \in S_+^{n-1}} S(x)$$

is $\mathcal{C}^\infty$-diffeomorphic to $S_+^{n-1} \times R$ and $S_+^{n-1} \times S$, where R and S are "model" manifolds for all $R(x)$'s and for all $S(x)$'s, respectively. Arguing further, analogously as above, we also infer that

$$Z(h) \cap R(x) \in \mathscr{I}_{R(x)} \quad \text{and} \quad Z(h) \cap S(x) \in \mathscr{I}_{S(x)} \quad \mathscr{I}_{S_+^{n-1}}\text{-(a.e.)}. \tag{16}$$

According to (15) and (16) there is a set $S_0 \in \mathscr{I}_{S_+^{n-1}}$ with

$$\begin{cases} Z(h) \cap Q(x) \in \mathscr{I}_{Q(x)}, \\ Z(h) \cap R(x) \in \mathscr{I}_{R(x)}, \\ Z(h) \cap S(x) \in \mathscr{I}_{S(x)} \end{cases} \tag{17}$$

for $x \in S_+^{n-1} \setminus S_0$.

At the moment, assume that $n = 2$. Applying Lemma 7 to the set

$$A := S_0 \cup (-S_0) \cup \{(-1, 0), (1, 0)\} \in \mathscr{I}_{S^1},$$

and changing signs of vectors of the obtained basis as required, we get an orthogonal basis $(x^{(1)}, x^{(2)})$ of $\mathbb{R}^2$ whose each element x satisfies conditions (17).

Now, we shall prove that for each $i \in \{1, 2\}$ the function $h_i : \mathbb{R} \to G$ given by $h_i(\lambda) = h(\lambda x^{(i)})$ satisfies

$$h_i(\lambda + \mu) = h_i(\lambda) + h_i(\mu) \quad \Omega(\mathscr{I}_{(0,\infty)})\text{-(a.e.)}, \tag{18}$$

where $\Omega(\mathscr{I}_{(0,\infty)}) = \{A \subset (0, \infty)^2 : A[x] \in \mathscr{I}_{(0,\infty)}\mathscr{I}_{(0,\infty)}\text{-(a.e.)}\}$ is the so called conjugate ideal. Plainly, condition (18) would imply that the same is true with $(0, \infty)$ replaced by $(-\infty, 0)$, due to the oddness of the function h.

Fix $i \in \{1, 2\}$. In view of (17), with x replaced by $x^{(i)}$, there is a set $C_i \in \mathscr{I}_{\mathbb{R}^* \times P^*_{x^{(i)}}}$ such that

$$\begin{cases} \left(\lambda x^{(i)} + y, \dfrac{\|y\|^2}{\lambda} x^{(i)} - y\right) \in \perp^* \setminus Z(h), \\ \left(\dfrac{\|y\|^2}{\lambda} x^{(i)}, -y\right) \in \perp^* \setminus Z(h), \\ \left(\lambda x^{(i)}, y\right) \in \perp^* \setminus Z(h) \end{cases} \tag{19}$$

for $(\lambda, y) \in (\mathbb{R}^* \times P^*_{x^{(i)}}) \setminus C_i$ (note that $T_{x^{(i)}} \in \mathscr{I}_{\mathbb{R}^* \times P^*_{x^{(i)}}}$, so we may include the set $T_{x^{(i)}}$ into C_i and we see that the difference between the domain of $\Phi_{x^{(i)}}$ and the domains of $\Gamma_{x^{(i)}}$, $\Theta_{x^{(i)}}$ causes no trouble at all). Therefore, for all $\lambda \in \mathbb{R}$ except a set $\Lambda_i \in \mathscr{I}_1$ the conjunction (19) holds true for all $y \in P_{x^{(i)}} \setminus Y_i(\lambda)$ with $Y_i(\lambda) \in \mathscr{I}_{P_{x^{(i)}}}$. Let

$$B_i(\lambda) = \left\{ \frac{\|y\|^2}{\lambda} : y \in P_{x^{(i)}} \setminus Y_i(\lambda) \right\}.$$

Then, obviously, $\mathbb{R} \setminus B_i(\lambda) \in \mathscr{I}_{(0,\infty)}$ for each positive $\lambda \notin \Lambda_i$, whereas $\mathbb{R} \setminus B_i(\lambda) \in \mathscr{I}_{(-\infty,0)}$ for each negative $\lambda \notin \Lambda_i$. For every pair (λ, μ) with $\lambda \notin \Lambda_i$ and $\mu \in B_i(\lambda)$, $\mu = \frac{\|y\|^2}{\lambda}$, we have

$$\begin{aligned} h_i(\lambda + \mu) &= h\left(\lambda x^{(i)} + y + \frac{\|y\|^2}{\lambda} x^{(i)} - y\right) = h(\lambda x^{(i)} + y) + h\left(\frac{\|y\|^2}{\lambda} x^{(i)} - y\right) \\ &= h(\lambda x^{(i)}) + h(y) + h\left(\frac{\|y\|^2}{\lambda} x^{(i)}\right) + h(-y) = h_i(\lambda) + h_i(\mu), \end{aligned}$$

which proves (18). Applying the theorem of de Bruijn [4] separately to the functions $h_i|_{(0,\infty)}$ and $h_i|_{(-\infty,0)}$ we get two additive mappings $b'_i : (0, \infty) \to G$ and $b''_i : (-\infty, 0) \to G$ which coincide with these two restrictions of h_i almost everywhere in $(0, \infty)$ and $(-\infty, 0)$, respectively. However, since h is odd, the extensions of both b'_i and b''_i to the whole real line have to be the same. As a result, there is an additive function $b_i : \mathbb{R} \to G$ such that $h_i(\lambda) = b_i(\lambda)$ for $\lambda \in \mathbb{R} \setminus Z_i$ with a certain $Z_i \in \mathscr{I}_1$.

Define a function $b : \mathbb{R}^2 \to G$ by $b(x) = b_1(\lambda_1) + b_2(\lambda_2)$, where λ_i is the ith coordinate of x with respect to the basis $(x^{(1)}, x^{(2)})$. Plainly, b is an additive function. It remains to show that $h(x) = b(x)\mathscr{I}_2$-(a.e.).

Recall that for every $x \in X = \mathbb{R} \times \mathbb{R}^*$ the mapping Ψ_x defined by (10) yields a $\mathcal{C}^\infty$-diffeomorphism between $\mathbb{R}^* \times P^*_x$ and $\mathbb{R}^* \times \mathbb{R}^*$. In particular, we have $C := \Psi_{x^{(1)}}(C_1) \in \mathscr{I}_2$ and

$$\left(\lambda x^{(1)}, \widetilde{\psi}^{-1}_{x^{(1)}}(y)\right) \in \perp^* \setminus Z(h) \quad \text{for } (\lambda, y) \in \mathbb{R}^2 \setminus C. \tag{20}$$

Define $\Delta : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Delta(\lambda_1, \lambda_2) = \left(\lambda_1, \widetilde{\psi}_{x^{(1)}}(\lambda_2 x^{(2)})\right).$$

Plainly, Δ is a $\mathcal{C}^\infty$-diffeomorphism, so $\Delta^{-1}(C) \in \mathscr{I}_2$. Therefore,

$$\Delta^{-1}(C) \cup (Z_1 \times \mathbb{R}) \cup (\mathbb{R} \times Z_2) \in \mathscr{I}_2$$

and for each pair $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ outside this set condition (20) implies $(\lambda_1 x^{(1)}, \lambda_2 x^{(2)}) \in \perp^* \setminus Z(h)$; thus

$$\begin{aligned} h(\lambda_1 x^{(1)} + \lambda_2 x^{(2)}) &= h(\lambda_1 x^{(1)}) + h(\lambda_1 x^{(1)}) = h_1(\lambda_1) + h_2(\lambda_2) \\ &= b_1(\lambda_1) + b_2(\lambda_2) = b(\lambda_1 x^{(1)} + \lambda_2 x^{(2)}). \end{aligned}$$

By the isomorphism, which to every $x \in \mathbb{R}^2$ assigns its coordinates in the basis $(x^{(1)}, x^{(2)})$, we have $h(x) = b(x)\mathscr{I}_2$-(a.e.) and our assertion for $n = 2$ follows.

In the sequel, assume that $n \geq 3$ and the assertion holds true for $n - 1$ in the place of n.

Define $\mathrm{O}(n - 1, n)'$ to be the set of all $(n - 1)$-tuples from $\mathrm{O}(n - 1, n)$ generating a subspace of $\mathbb{R}^n$ whose orthogonal complement is spanned by a vector $(x_1, \ldots, x_n)$ with $x_n \neq 0$. In other words,

$$\mathrm{O}(n - 1, n)' = \left\{ (x^{(1)}, \ldots, x^{(n-1)}) \in \mathrm{O}(n - 1, n) : \pm \frac{x^{(1)} \wedge \cdots \wedge x^{(n-1)}}{\|x^{(1)} \wedge \cdots \wedge x^{(n-1)}\|} \in S^{n-1}_+ \right\},$$

where $\wedge$ stands for the wedge product in $\mathbb{R}^n$. This set, being an open subset of $\mathrm{O}(n-1,n)$, is its submanifold having the same dimension. Consider the mapping $\Omega\colon S_+^{n-1}\times \mathrm{O}(n-1,n-1)\to \mathrm{O}(n-1,n)'$ defined by

$$\Omega(x,x^{(1)},\dots,x^{(n-1)})=\big(\widetilde{\psi}_x^{-1}(x^{(1)}),\dots,\widetilde{\psi}_x^{-1}(x^{(n-1)})\big).$$

The values of Ω indeed belong to $\mathrm{O}(n-1,n)'$, since for each $x\in X$ the function ψ_x is an isometry, being a linear map determined by the orthogonal matrix $\boldsymbol{Y}(x)^{-1}$. Furthermore, Ω is bijective with the inverse Ω^{-1} given by

$$\Omega^{-1}(y^{(1)},\dots,y^{(n-1)})=\big(x,\widetilde{\psi}_x(y^{(1)}),\dots,\widetilde{\psi}_x(y^{(n-1)})\big),$$

where

$$x=\pm\frac{y^{(1)}\wedge\cdots\wedge y^{(n-1)}}{\|y^{(1)}\wedge\cdots\wedge y^{(n-1)}\|}$$

and the sign depends on which of the two components of $\mathrm{O}(n-1,n)'$ contains $(y^{(1)},\dots,y^{(n-1)})$. By the above formulas, Ω is a $\mathcal{C}^\infty$-diffeomorphism.

Put

$$Z=\{(y^{(1)},\dots,y^{(n-1)})\in \mathrm{O}(n-1,n)' : (y^{(1)},y^{(2)})\in Z(h)\}.$$

Then Lemma 5 implies $Z\in\mathscr{I}_{\mathrm{O}(n-1,n)'}$, since $Z(h)\in\mathscr{I}_\perp$ (i.e. $Z(h)\in\mathscr{I}_{\mathrm{O}(2,n)}$) is the image of Z through the $\mathcal{C}^\infty$-submersion $(y^{(1)},\dots,y^{(n-1)})\mapsto (y^{(1)},y^{(2)})$. Therefore, we have $\Omega^{-1}(Z)\in\mathscr{I}_{S_+^{n-1}\times \mathrm{O}(n-1,n-1)}$; hence $\Omega^{-1}(Z)[x]\in\mathscr{I}_{\mathrm{O}(n-1,n-1)}$ is valid $\mathscr{I}_{S_+^{n-1}}$-(a.e.), which translates into the fact that the set

$$A(x):=\Big\{(x^{(1)},\dots,x^{(n-1)})\in \mathrm{O}(n-1,n-1) : \big(\widetilde{\psi}_x^{-1}(x^{(1)}),\widetilde{\psi}_x^{-1}(x^{(1)})\big)\in Z(h)\Big\}$$

belongs to $\mathscr{I}_{\mathrm{O}(n-1,n-1)}$ for every $x\in S_+^{n-1}$ except a set from $\mathscr{I}_{S_+^{n-1}}$. By virtue of Lemma 9, for each such x we must have

$$\Big\{(x^{(1)},x^{(2)})\in \mathrm{O}(2,n-1) : \big(\widetilde{\psi}_x^{-1}(x^{(1)}),\widetilde{\psi}_x^{-1}(x^{(2)})\big)\in Z(h)\Big\}\in\mathscr{I}_{\mathrm{O}(2,n-1)}. \tag{21}$$

Hence, putting $\perp_x=\{(t,u)\in P_x\times P_x : (t,u)\in\perp\}$ we infer that the condition

$$h(t+u)=h(t)+h(u)\quad \mathscr{I}_{\perp_x^*}\text{-(a.e.)} \tag{22}$$

is valid $\mathscr{I}_{S_+^{n-1}}$-(a.e.). Consequently, we may pick a particular $x\in S_+^{n-1}$ satisfying both (17) and (22). By virtue of our inductive hypothesis and some isometry formalities (identifying P_x with $\mathbb{R}^{n-1}$), condition (22) yields the existence of an additive function $b_x\colon P_x\to G$ such that $h(t)=b_x(t)$ for $t\in P_x\setminus Y$ with a certain $Y\in\mathscr{I}_{P_x}$. Moreover, by an earlier argument, there is also an additive function $b_1\colon\mathbb{R}\to G$ such that $h(\lambda x)=b_1(\lambda)$ for $\lambda\in\mathbb{R}\setminus Z_1$ with a certain $Z_1\in\mathscr{I}_1$. Finally, there is a set $C_1\in\mathscr{I}_{\mathbb{R}\times P_x}$ with $(\lambda x,y)\in\perp^*\setminus Z(h)$ whenever $(\lambda,y)\in(\mathbb{R}\times P_x)\setminus C_1$.

Define a function $b\colon\mathbb{R}^n\to G$ by the formula $b(\lambda x+y)=b_1(\lambda)+b_x(y)$ for $\lambda\in\mathbb{R}$ and $y\in P_x$. Then b is additive and for each pair $(\lambda,y)\in\mathbb{R}\times P_x$ outside the set

$$C_1\cup(Z_1\times P_x)\cup(\mathbb{R}\times Y)\in\mathscr{I}_{\mathbb{R}\times P_x}$$

we have

$$h(\lambda x+y)=h(\lambda x)+h(y)=b_1(\lambda)+b_x(y)=b(\lambda x+y),$$

which completes the proof. $\square$

Lemma 11. *If a function $h\colon\mathbb{R}^n\to G$ satisfies $h(x)=h(-x)\,\mathscr{I}_n$-(a.e.) and $h(x+y)=h(x)+h(y)$ $\mathscr{I}_\perp$-(a.e.), then there is an additive function $a\colon\mathbb{R}\to G$ such that $h(x)=a\big(\|x\|^2\big)$ $\mathscr{I}_n$-(a.e.).*

Proof. For any $r\geq 0$ let $S^{n-1}(r)=\{x\in\mathbb{R}^n : \|x\|=r\}$. By the natural identification, we have $(\mathbb{R}^n)^*\sim(0,\infty)\times S^{n-1}$. Therefore, for every $A\in\mathscr{I}_n$ there is a set $R(A)\in\mathscr{I}_{(0,\infty)}$ such that $A\cap S^{n-1}(r)\in\mathscr{I}_{S^{n-1}(r)}$ for $r\in(0,\infty)\setminus R(A)$. In the first part of the proof we will show the following claim: there exists a set $A\in\mathscr{I}_n$ such that for each $r\in(0,\infty)\setminus R(A)$ the function h is constant $\mathscr{I}_{S^{n-1}(r)}$-(a.e.) on $S^{n-1}(r)$, more precisely—that $h|_{S^{n-1}(r)}$ is constant outside the set $A\cap S^{n-1}(r)$.

We start with the following observation: there is $T\in\mathscr{I}_\perp$ such that $h(t+u)=h(u-t)$ whenever $(t,u)\in\perp^*\setminus T$. Let $E=\{x\in\mathbb{R}^n : h(x)=h(-x)\}$ and $H=(-D(h))\cap D(h)\cap E$; then $\mathbb{R}^n\setminus H\in\mathscr{I}_n$. Define

$$T=\{(t,u)\in\perp^* : t\notin H\}\cup\{(t,u)\in\perp^* : t\in H \text{ and } u\notin E_t(h)\cap E_{-t}(h)\}. \tag{23}$$

Then for every $(t,u)\in\perp^*\setminus T$ we have $h(t+u)=h(t)+h(u)$ and $h(u-t)=h(u)+h(-t)$. Moreover, we have also $h(t)=h(-t)$; hence $h(t+u)=h(u-t)$, as desired. In order to show that $T\in\mathscr{I}_\perp$ note that it is equivalent to $T\cap\perp'\in\mathscr{I}_{\perp'}$, where $\perp'$ may be identified with $X\times\mathbb{R}^{n-1}$. The first summand in (23), after intersecting with $\perp'$, is then identified with $(X \setminus H) \times \mathbb{R}^{n-1} \in \mathscr{I}_{2n-1}$, whereas for each pair (t, u) from the second summand we have either $(t, u) \in Z(h)$, or $(-t, u) \in Z(h)$, which shows that it belongs to $\mathscr{I}_\perp$. Consequently, $T \in \mathscr{I}_\perp$.

Define $\Phi: \perp^* \to \mathbb{R}^n \times \mathbb{R}^n$ by putting $\Phi(t, u) = (t + u, u - t)$. It is evident that Φ is a $\mathcal{C}^\infty$-immersion and yields a homeomorphism between $\perp^*$ and

$$M := \Phi(\perp^*) = \bigcup_{r\in(0,\infty)} (S^{n-1}(r) \times S^{n-1}(r)).$$

Therefore, [12, Theorem 11.17] implies that M is a manifold. Moreover, $\Phi: \perp^* \to M$ is a $\mathcal{C}^\infty$-diffeomorphism; thus $\Phi(T) \in \mathscr{I}_M$. Since the mapping $(x, y) \mapsto (x, y/\|x\|)$ yields $M \sim (\mathbb{R}^n)^* \times S^{n-1}$, there exists a set $A \in \mathscr{I}_n$ such that for every $x \in \mathbb{R}^n \setminus A$ we have

$$(x, y) \notin \Phi(T) \quad \mathscr{I}_{S^{n-1}(\|x\|)}\text{-(a.e.)}.$$

By the property of the set T, $(x, y) \notin \Phi(T)$ implies $h(x) = h(y)$. Now, for any $r \in (0, \infty) \setminus R(A)$ and for arbitrary $x, y \in \mathbb{R}^n \setminus A$ with $\|x\| = \|y\| = r$, we have

$$(x, z), (y, z) \notin \Phi(T) \quad \mathscr{I}_{S^{n-1}(r)}\text{-(a.e.)};$$

hence $h(x) = h(z) = h(y)$, which completes the proof of our claim.

There is a function $g: \mathbb{R}^n \to G$ which is constant on every sphere $S^{n-1}(r)$ such that $h(x) = g(x)$ for $x \in \mathbb{R}^n \setminus A$. Therefore, there is also a function $\varphi: [0, \infty) \to G$ satisfying $g(x) = \varphi\left(\|x\|^2\right)$ for every $x \in \mathbb{R}^n$. We are going to show that

$$\varphi(\lambda + \mu) = \varphi(\lambda) + \varphi(\mu) \quad \Omega(\mathscr{I}_{(0,\infty)})\text{-(a.e.)}. \tag{24}$$

Put

$$B = \{(x, y) \in \perp^* : \text{ either } x \in A, \text{ or } y \in A, \text{ or } x + y \in A\}$$

and observe that $B \in \mathscr{I}_\perp$, whence also $Z := Z(h) \cup B \in \mathscr{I}_\perp$. Let

$$D = \{x \in (\mathbb{R}^n)^* : (x, y) \notin Z \ \mathscr{I}_{P_x}\text{-(a.e.)}\}.$$

By an argument similar to the one applied to $D(h)$, we infer that $X \setminus D \in \mathscr{I}_X$; hence $\mathbb{R}^n \setminus D \in \mathscr{I}_n$. For each $x \in \mathbb{R}^n$ put $E_x = \{y \in P_x : (x, y) \notin Z\}$; then $P_x \setminus E_x \in \mathscr{I}_{P_x}$ provided $x \in D$. Let also $D' = \{\|x\|^2 : x \in D\}$; then $(0, \infty) \setminus D' \in \mathscr{I}_{(0,\infty)}$.

Fix arbitrarily $\lambda \in D'$ and choose any $x \in D$ satisfying $\sqrt{\lambda} = \|x\|$. Put $E(\lambda) = \{\|y\|^2 : y \in E_x\}$ (then $(0, \infty)\setminus E(\lambda) \in \mathscr{I}_{(0,\infty)}$) and pick any $\mu \in E(\lambda)$. Then $\sqrt{\mu} = \|y\|$ for some $y \in E_x$, which implies $(x, y) \notin Z$. Applying the facts that $x+y \notin A$, $(x, y) \notin Z(h)$, $x \notin A$ and $y \notin A$, consecutively, we obtain

$$\begin{aligned}\varphi(\lambda + \mu) &= g(x + y) = h(x + y)\\ &= h(x) + h(y) = g(x) + g(y) = \varphi(\lambda) + \varphi(\mu),\end{aligned}$$

which proves (24).

By the theorem of de Bruijn, there is an additive function $a: \mathbb{R} \to G$ such that $\varphi(\lambda) = a(\lambda)$ for $\lambda \in [0, \infty) \setminus Y$ with $Y \in \mathscr{I}_{[0,\infty)}$. Then the equality $h(x) = a\left(\|x\|^2\right)$ holds true for $x \in \mathbb{R}^n \setminus (A \cup C)$, where $C = \{x \in \mathbb{R}^n : \|x\|^2 \in Y\} \in \mathscr{I}_n$. Thus, the proof has been completed. □

To finish the proof of our theorem we shall combine Lemmas 8, 10 and 11 to get additive functions $a: \mathbb{R} \to G$ and $b: \mathbb{R}^n \to G$ such that

$$2\left(f(x) - a(\|x\|^2) - b(x)\right) = 0 \quad \mathscr{I}_n\text{-(a.e.)}.$$

The only thing left to be proved is the following fact in the spirit of [2, Lemma 2].

Lemma 12. *If a function $h: \mathbb{R}^n \to G$ satisfies $2h(x) = 0$ $\mathscr{I}_n$-(a.e.) and $h(x+y) = h(x)+h(y)$ $\mathscr{I}_\perp$-(a.e.), then $h(x) = 0$ $\mathscr{I}_n$-(a.e.).*

Proof. For every $x \in \mathbb{R}^n$ put $g(x) = h(x) - h(-x)$. Applying Lemmas 8 and 10 we get an additive function $b: \mathbb{R}^n \to G$ such that $g(x) = b(x)\mathscr{I}_n$-(a.e.). Therefore

$$g(x) = 2b\left(\frac{x}{2}\right) = 2h\left(\frac{x}{2}\right) - 2h\left(-\frac{x}{2}\right) = 0 \quad \mathscr{I}_n\text{-(a.e.)},$$

i.e. $h(x) = h(-x)\mathscr{I}_n$-(a.e.). Now, by virtue of Lemma 11, there is an additive function $a: \mathbb{R} \to G$ satisfying $h(x) = a(\|x\|^2)\mathscr{I}_n$-(a.e.). Consequently,

$$h(x) = a\left(2\left\|\frac{1}{\sqrt{2}}x\right\|^2\right) = 2a\left(\left\|\frac{1}{\sqrt{2}}x\right\|^2\right) = 2h\left(\frac{1}{\sqrt{2}}x\right) = 0 \quad \mathscr{I}_n\text{-(a.e.)}. \quad \square$$

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